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Pierre DE LA HARPE and Claude WEBER

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ON MALNORMAL PERIPHERAL SUBGROUPS OF THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

PIERRE DE LA HARPE AND CLAUDE WEBER

Abstract. Let K be a non-trivial knot in the 3-sphere, E_K its exterior, $G_K = \pi_1(E_K)$ its group, and $P_K = \pi_1(\partial E_K) \subset G_K$ its peripheral subgroup. We show that P_K is malnormal in G_K, namely that $gP_Kg^{-1} \cap P_K = \{e\}$ for any $g \in G_K$ with $g \notin P_K$, unless K is in one of the following three classes: torus knots, cable knots, and composite knots; these are exactly the classes for which there exist annuli in E_K attached to T_K which are not boundary parallel (Theorem 1 and Corollary 2). More generally, we characterise malnormal peripheral subgroups in the fundamental group of a compact orientable irreducible 3-manifold of which the boundary is a non-empty union of tori (Theorem 3). Proofs are written with non-expert readers in mind. Half of our paper (Appendices A to D) is a reminder of some three-manifold topology as it flourished before the Thurston revolution.

In a companion paper [15], we collect general facts on malnormal subgroups and Frobenius groups, and we review a number of examples.

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1. Statement of the results

Consider a knot K in $\mathbf{S}^3$. Let V_K be a tubular neighbourhood of K. The **exterior** of K is the closure E_K of $\mathbf{S}^3 \smallsetminus V_K$, and the **peripheral torus** is the common boundary $T_K = \partial V_K = \partial E_K$. The **group of** K is the fundamental group $G_K = \pi_1(E_K)$, and the **peripheral subgroup** is the image P_K of $\pi_1(T_K)$ in G_K. Recall that, by Dehn's Lemma, the map $\pi_1(T_K) \longrightarrow P_K$ is an isomorphism if and only if K is non-trivial.

A subgroup H of a group G is **malnormal** if $gHg^{-1} \cap H = \{e\}$ for all $g \in G$ with $g \notin H$; basic facts on malnormal subgroups can be found in our companion paper [15]. The following question arose in discussions with Rinat Kashaev (see also [21] and [22]); we are grateful to him for this motivation.

Given K as above, when is P_K malnormal in G_K ?

The answer, our Corollary 2, happens to be a straightforward consequence of the following Theorem, from [32]; the latter appears already as Lemma 1.1 in [38], and also as Proposition 2 in [12]. Our proof, essentially self-contained, relies on Seifert foliations and pseudo-foliations. Technical terms are defined below (see Sections 2, 3, and the three appendices A, B, and C).

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Theorem 1 (Reformulation of a result of Jonathan Simon). — *Let K be a non-trivial knot, E_K its exterior, T_K its boundary, and μ a meridian of T_K. Assume that there exists an annulus A in E_K attached to T_K which is not boundary parallel. Then the knot K is*

(i) *either a composite knot,*
(ii) *or a torus knot,*
(iii) *or a cable knot.*

Moreover, let η denote one of the components of ∂A. In case (i) η is a meridian of T_K. In cases (ii) and (iii), the distance of μ and η is $\Delta(\mu, \eta) = 1$. In particular, in all cases, $\Delta(\mu, \eta) \leqslant 1$.

Conversely, if K is as in one of (i), (ii), and (iii), then there exists an annulus A in E_K attached to T_K which is not boundary parallel.

The "converse part" of the theorem is a rather straightforward consequence of the definitions, see Section 3. As a corollary of Theorem 1 and the annulus theorem:

Corollary 2. — *For a non-trivial knot K, the peripheral group P_K is malnormal in G_K if and only if K is neither a composite knot, nor a torus knot, nor a cable knot.*

To view P_K as a subgroup of G_K, we need to choose a path from the base point in E_K implicitely used to define G_K to the base point in T_K implicitely used to define $\pi_1(T_K)$, so that P_K is a subgroup of G_K defined up to conjugation only. But the conclusion of Corollary 2 makes sense since a subgroup and all its conjugates are together malnormal or not. A similar remark holds for the next theorem.

Theorem 1 suggests a result on malnormal peripheral subgroups in a more general situation:

Theorem 3. — *Let M be a 3-manifold which is compact, connected, orientable, and irreducible. Assume moreover that the boundary ∂M has at least one component, say $\partial_1 M$, which is a torus; and that M is neither a solid torus $\mathbf{S}^1 \times \mathbf{D}^2$ nor a thickened torus $\mathbf{T}^2 \times [0, 1]$.*

Denote by G the fundamental group of M, by P the peripheral subgroup of G associated with $\partial_1 M$, and by V the connected component of the Jaco-Shalen-Johannson decomposition of M which contains $\partial_1 M$.

Then P is not malnormal in G if and only if V is a Seifert manifold.

Two observations are in order.

If M is a solid torus or a thickened torus (in both cases an irreducible Seifert manifold), the peripheral group coincides with $\pi_1(M)$, and thus is trivially malnormal in $\pi_1(M)$.

There is a well-known fact on 3-manifolds which are compact, connected, orientable, irreducible, and with non-empty boundary: **if one boundary component of such a manifold is a compressible torus, then the manifold is a solid torus;** for the convenience of the reader, we provide a proof of this as Lemma 13 below. Thus in the situation of Theorem 3, $\partial_1 M$ is incompressible in M.

By specialising to exteriors of links (see the definitions recalled at the end of Appendix B), we could obtain the following corollary. The notation we use for a link L, namely V_L, E_L and G_L, are defined in the same way as for knots.

Corollary 4. — *Let L be a link in $\mathbf{S}^3$, with $r \geqslant 2$ components $L_1, \ldots, L_r$. Assume that L is unsplittable, and is not the Hopf link. Denote by G_L the group of L. For $j \in \{1, \ldots, r\}$, denote by P_j the peripheral subgroup of G_L which corresponds to L_j.*

Then P_j is not malnormal in G_L if and only if

- *either L_j is part of a (possibly satellised) torus sublink of L;*
- *or L_j is the outcome of a connected sum operation of links.*

In this paper, we give a proof of Theorem 1, following the method of [32]; from this and the annulus theorem, Corollary 2 follows. Then, using more of the theory of 3-manifolds, we prove Theorem 3; as a consequence, we obtain a second proof of Theorem 1.

More precisely, Section 2 contains general facts on annuli attached to boundary tori of 3-manifolds. Section 3 analyses exteriors of composite knots, torus knots, and cable knots, and thus establishes the converse (and easy) implication in Theorem 1. In Section 4, we complete the proof of Theorem 1; the first few lines show also how Corollary 2 follows from Theorem 1. Section 5 is a proof of Theorem 3, and Section 6 shows how Corollary 2 follows from Theorem 3.

We will not show how Corollary 4 follows from Theorem 3, for length reasons. Indeed, if $T_1, \dots, T_r$ denote the boundary components of the exterior E_L of the link $L = L_1 \sqcup \cdots \sqcup L_r$, a JSJ piece of E_L can be adjacent to just one of the T_j or to several of them, and many cases have to be treated separately, so that there are (among other things) non-trivial combinatorial complications. To avoid unreasonable length, we have chosen to leave the details to the expert readers.

As we have non-expert readers in mind, we have written rather long appendices. In A, we recall various basic definitions on 3-manifolds, a theorem due to Alexander on complements of tori in $\mathbf{S}^3$, and a re-embedding construction of Bonahon and Siebenmann for submanifolds of $\mathbf{S}^3$. Appendix B is about Seifert foliations and Seifert pseudo-foliations on 3-manifolds. Appendix C is a reminder on the annulus theorem and the JSJ decomposition, needed for our proof of Theorem 3. The last appendix is a digression on the terminology and the literature.

In a companion paper [15], we collect basic facts and (more or less) standard examples on malnormal subgroups and on Frobenius groups of permutations.

It is convenient to agree on the following **standing assumption**:

all 3*-manifolds and surfaces below are assumed to be*
compact, connected, orientable, and possibly with boundary,

unless either they are obviously not, such as links or boundaries, which need not be connected, or if it is explicitely stated otherwise (for example, the space of leaves of a Seifert foliation, a surface, need not be orientable). Moreover, maps, and in particular embeddings from one manifold into another, are assumed to be **smooth**.

2. On annuli embedded in 3-manifolds with torus boundaries

2.1. **Curves in tori and slopes.** A simple closed curve in a surface is **essential** if it is not homotopic to a point, equivalently if it does not bound an embedded disc. Let T be a 2-dimensional torus; a **slope** in T is an isotopy class of essential simple closed curves. These curves and slopes are non-oriented.

The **distance** $\Delta(s_1, s_2)$ of two slopes s_1, s_2 in T is the absolute value of their intersection number, namely

$$\Delta(s_1, s_2) \,=\, \min \left\{ \#(\sigma_1 \cap \sigma_2) \;\middle|\; \begin{array}{l} \sigma_j \text{ is a simple closed curve} \\ \text{representing } s_j,\ j = 1, 2 \end{array} \right\}.$$

(This "distance" does not satisfy the triangle inequality, but the terminology is however standard.) Two slopes are isotopic if and only if their distance is zero. Two slopes, once oriented, define a basis of $H_1(T, \mathbf{Z})$ if and only if their distance is one. Observe that, if σ_1, σ_2 are two essential simple closed curves in T which are disjoint, and therefore isotopic, then the closure of each connected component of their complement $T \smallsetminus (\sigma_1 \cup \sigma_2)$ is an annulus embedded in T.

For curves on tori and for slopes, see [27] (in particular Section 2.C) and [5].

If a 2-torus T is given as the boundary of a solid torus, a **meridian** is an essential simple closed curve μ on T which bounds a disc in the solid torus, and a **parallel** is an essential simple closed curve λ on T such that the homotopy classes of λ and μ, with orientations, constitute a basis of $\pi_1(T) = H_1(T, \mathbf{Z})$.

2.2. **Annuli attached to a torus component of the boundary.** Let M be a 3-manifold with boundary, such that one connected component of ∂M, say T, is a 2-torus. An **annulus in** M **attached to** T is an annulus A which is properly embedded in M and such that each component of ∂A is an essential curve in T; observe that these two components are disjoint, so that we have a well-defined **slope of** A **in** T.

As a particular case of a definition from Subsection A.1, an annulus A in M attached to $T \subset \partial M$ is **boundary parallel** if there exists a solid torus U embedded in M such that

(i) $A \subset \partial U$,
(ii) $(\partial U \smallsetminus A) \subset T$,
(iii) there exists a diffeomorphism[1] $h : U \longrightarrow A \times [0, 1]$ such that $h(A) = A \times \{0\}$;

in this case A is said to be boundary parallel **through** U. Observe that, in this situation, there is an annulus A_T embedded in T such that $\partial U = A \cup_{\partial A} A_T$ (the notation $\cup_{\partial A}$ indicates that $A \cap A_T = \partial A$).

Our next Subsections, 2.3 to 3.3, describe various examples in the particular situation of the exterior of a knot.

2.3. **Examples of boundary parallel annuli in knot exteriors.** Let K be a knot; let E_K and $T_K = \partial E_K$ be as usual. Any slope s in T_K can be the slope of a boundary parallel torus attached to T_K in E_K.

Indeed, consider an annulus A_T in T_K bounded by two parallel essential simple closed curves in T_K in the class s. Push the interior of A_T slightly inside E_K to obtain an annulus A in M attached to T_K; there is a well-defined solid torus U, bounded by $A \cup A_T$, such that A is boundary parallel through U.

(This would essentially carry over to any boundary component of any 3-manifold, instead of just the torus ∂E_K.)

On the contrary, Theorem 1 shows that there are strong limitations on slopes which can be associated to non-boundary parallel annuli attached to T_K. More generally there are strong limitations on slopes of incompressible surfaces, see e.g. [5].

2.4. **Examples of annuli in the exterior of the trivial knot.** Consider a non-trivial knot J, with E_J, T_J as usual, and a meridian $K \subset T_J$, which is viewed as a trivial knot in $\mathbf{S}^3$, with E_K, T_K as usual. Then $A := T_J \cap E_K$ is an annulus in E_K attached to T_K. It is not boundary parallel because J is non-trivial. The slope of A in T_K is a parallel.

3. Examples of non boundary-parallel annuli in knot exteriors

The next three subsections provide a proof of the converse part of Theorem 1.

3.1. **Examples of annuli in exteriors of composite knots.** Consider a composite knot, namely a connected sum $K = K_1 \sharp K_2$ of two non-trivial knots. There is no loss of generality if we assume that K is in $\mathbf{R}^3$, intersecting $\mathbf{R}^2$ in exactly two points, and that, if H_1, H_2 denote the two closed half-spaces bounded by $\mathbf{R}^2$, the knot K_j is the union of $K \cap H_j$ with the straight segment in $\mathbf{R}^2$ joining the two points of $K \cap \mathbf{R}^2$ (for $j = 1, 2$). Then

$$A = \big(\mathbf{R}^2 \smallsetminus (\mathbf{R}^2 \cap \overset{\circ}{V}_K)\big) \cup \{\infty\}$$

is an annulus in E_K attached to T_K. The slope of A in T_K is a meridian.

For $j = 1, 2$, denote by W_j the closure of the complement in H_j of $H_j \cap V_K$. Observe that W_j is diffeomorphic to the exterior of K_j so that both W_1 and W_2 are knot exteriors.

[1] Note that U and $A \times [0, 1]$ are manifolds *with corners*; the notion of diffeomorphism can easily be adapted to this situation.

(Note: if K_2, say, were trivial, then W_2 would be a solid torus.)

3.2. **Examples of annuli in exteriors of torus knots.** Denote by $\mathbf{S}^1 \times \mathbf{D}^2$ the solid torus standardly embedded in $\mathbf{R}^3$ and by $\mathbf{T}$ its boundary. A **torus knot** is a knot isotopic to an essential simple closed curve on the torus $\mathbf{T}$. We agree here that

the trivial knot is not a torus knot.

Let K be a torus knot on $\mathbf{T}$, and let V_K be a tubular neighbourhood of K small enough for $V_K \cap \mathbf{T}$ to be a pair (K_1, K_2) of curves on $\mathbf{T}$ which are disjoint and parallel to K (the standardly embedded torus $\mathbf{T}$ should not be confused with the torus $T_K = \partial V_K = \partial E_K$). The complement $\mathbf{T} \smallsetminus (K_1 \cup K_2)$ has two connected components; let A be the closure of the component which does not contain K. Then A is an annulus in $E_K = (\mathbf{R}^3 \cup \{\infty\}) \smallsetminus \mathring{V}_K$ attached to the boundary T_K. The slope of A in T_K is a parallel.

The complement $E_K \smallsetminus A$ of A has two connected components. The bounded component is essentially the interior of the standard solid torus; more precisely it is the interior of this standard solid torus minus part of the "small" solid torus V_K; thus, the closure of this bounded component is again a solid torus. Similarly, the other component, together with the point at infinity of $\mathbf{R}^3$, is a solid torus.

3.3. **Examples of annuli in exteriors of cable knots.** Consider on the one hand a non-trivial knot K_c and a tubular neighbourhood V_c of K_c with its boundary $T_c = \partial V_c$. Consider on the other hand the standardly embedded solid torus $\mathbf{S}^1 \times \mathbf{D}^2$, a non-trivial torus knot K_{pat} in $\partial(\mathbf{S}^1 \times \mathbf{D}^2)$, and a homeomorphism $h : \mathbf{S}^1 \times \mathbf{D}^2 \longrightarrow V_c$. Then, by definition, the knot $K := h(K_{pat})$ is a **cable knot around** K_c, with **companion** K_c and **pattern** K_{pat}. We do assume that K_c is non-trivial; thus, in this paper,

torus knots are not cable knots.

Some authors (including [32]) use the other convention, and consequently state Theorem 1 with two cases only.

Let A_{pat} be an annulus inside $\partial(\mathbf{S}^1 \times \mathbf{D}^2)$ related to K_{pat} as A is related to K in the previous Subsection 3.2. Then $A := h(A_{pat})$ is an annulus in E_K attached to the boundary T_K. The slope of A in T_K is again a parallel.

The two components of $E_K \smallsetminus A$ are one a solid torus (that is a small perturbation of V_c), and the other a knot exterior (a small perturbation of the exterior of K_c).

4. Proof of Theorem 1

We continue with the notation of Theorem 1. It is useful to consider a thickened torus

$$N_T := T_K \times [0, \epsilon] \subset V_K \quad \text{with} \quad N_T \cap E_K = T_K = T_K \times \{0\},$$

as well as a thickened annulus embedded in E_K

$$N_A := A \times [1, 2] \subset E_K \qquad \text{with} \quad A = A \times \{\frac{3}{2}\}.$$

Define

- the shrunk neighbourbood $V_K^- := V_K \smallsetminus N_T$,
- the enlarged exterior $E_K^+ := E_K \cup N_T = \overline{\mathbf{S}^3 \smallsetminus V_K^-}$,
- and their common boundary $T_K' := T_K \times \{\epsilon\} = \partial V_K^- = \partial E_K^+$, that is T_K slightly pushed inside V_K.

The union

$$N := N_T \cup N_A$$

is a manifold with boundary, indeed with corners. Note that

$$C_K := \partial A \times [1, 2] = T_K \cap N_A = N_T \cap N_A = V_K \cap N_A$$

is the disjoint union of two annuli, each one being a neighbourhood in T_K of a component of ∂A. On N, there is a natural foliation by circles, such that $\partial A \times \{1,2\}$ is the union of four particular leaves, of which the isotopy class is a slope of T_K. The manifolds N and N_A are irreducible since they are Seifert manifolds with boundary (Proposition 19).

Let Θ denote the space of leaves of N. Then Θ is homeomorphic to the union of an annulus (the space of leaves of N_T) together with a thickened diameter (the space of leaves of N_A); the four points which are common to the boundary of the thickened annulus and the boundary of the thickened diameter represent the four leaves in $\partial A \times \{1,2\}$. Since Θ is orientable (Lemma 26), it is a planar surface with three boundary components (a "pair of pants"), and the manifold N is diffeomorphic to a product:

$$N \approx \Theta \times \mathbf{S}^1.$$

The boundary ∂N is the union of three tori. One is T'_K; we denote the two others by T_1 and T_2. For $j \in \{1,2\}$, the torus T_j separates $\mathbf{S}^3$ in two components; we denote by W_j the closure of the component contained in E_K. Thus

$$E_K^+ = N \cup W_1 \cup W_2 \quad \text{and} \quad E_K = N_A \cup W_1 \cup W_2$$

where the interiors on the right-hand sides are disjoint.

By Alexander's Theorem 14, each W_j can be either a solid torus or a knot exterior, so that there are three cases to consider:

(4.1) both W_1 and W_2 are knot exteriors;
(4.2) both W_1 and W_2 are solid tori;
(4.3) W_1 is a solid torus and W_2 is a knot exterior.

We will see that these three cases correspond respectively to K being a composite knot, a torus knot, and a cable knot. Thus the proof below splits naturally in three cases; it follows and extends the indications given by [32] (there is an earlier paper of Simon with similar ingredients, used for other purposes).

4.1. **Case in which both W_1 and W_2 are knot exteriors.** We have to show that the slope of A in T_K is a meridian, and it will follow that *K is a connected sum of two non-trivial knots*. Compare with Subsection 3.1.

Since W_1 is a knot exterior, the manifold $\mathbf{S}^3 \setminus \overset{\circ}{W}_1$ is a solid torus, by Theorem 14. Thus W_2 is contained in the interior of a 3-ball which is contained in $\mathbf{S}^3 \setminus \overset{\circ}{W}_1$, by Proposition 16; we denote by Σ the boundary of this ball. Since $\Sigma \cap (W_1 \cup W_2)$ is empty, we have

$$\Sigma \subset \text{ interior of } (V_K \cup N_A)$$

and Σ separates W_1 from W_2. Since $V_K \cap N_A = C_K$, we have

$$\Sigma \cap \overset{\circ}{C}_K \neq \emptyset.$$

Indeed, if this intersection were empty, we would have either $\Sigma \subset V_K$ or $\Sigma \subset N_A$, and each of these inclusions would contradict the fact that Σ separates W_1 from W_2. Moving slightly Σ if necessary, we can assume that the intersection $\Sigma \cap C_K$ is transverse, so that this intersection, say Γ, is a bunch of pairwise disjoint circles.

We will show how to modify Γ (by modifying Σ), in order to obtain a curve which is both a slope of A in T_K and a meridian of T_K.

Consider a circle γ of the bunch Γ which is **innermost** in Σ, namely which bounds a disc, say $D_\gamma \subset \Sigma$, such that $\overset{\circ}{D}_\gamma \cap \Gamma = \emptyset$. There are two cases to consider, depending on this disc being in V_K or in N_A.

Suppose first that $D_\gamma \subset N_A$. Since the circle γ is in one of the two annuli making up $C_K = \partial A \times [1,2]$, there are a priori two possibilities: either it bounds a disc in this annulus, or it is parallel to the boundary of this annulus. But the second case would mean that γ defines the same slope of T_K as A; thus γ would be essential in T_K; since γ bounds a disc in E_K, the knot K would be trivial, in contradiction with our hypothesis. Hence γ bounds a disc $D'_\gamma \subset \partial A \times \{j\}$, where $j = 1$ or $j = 2$. The union $D_\gamma \cup D'_\gamma$ is a 2-sphere embedded in N_A; *since N_A is irreducible,* this 2-sphere bounds a 3-ball in N_A. We can now isotope D_γ through this 3-ball and then push it slightly outside N_A, to remove the intersection γ.

Suppose now that $D_\gamma \subset V_K$. As before, there are a priori two possibilites: either γ bounds a disc in C_K, and we can modifiy the situation to one with one circle less, or γ is both a slope of A and a meridian of T_K.

Iterating the previous construction with an innermost circle as often as necessary, we obtain in all cases a curve which is both a slope of A and a meridian of T_K. This ends the proof of Theorem 1 in case both W_1 and W_2 are knot exteriors.

4.2. **Case in which both W_1 and W_2 are solid tori.** There are *a priori* two subcases.

Either the oriented foliation of N introduced above extends to an oriented foliation by circles of $E_K^+ = N \cup W_1 \cup W_2$. Then K is a torus knot, by Proposition 28.

Or the oriented foliation of M does not extend to one of the W_j, say to W_1. Then, by Corollary 25, this foliation extends to W_2, indeed to $\mathbf{S}^3 \smallsetminus \overset{\circ}{W}_1$, with K a regular leaf, and the knot K is trivial. Since the triviality of K contradicts the hypotheses of Theorem 1, this second subcase does not occur.

4.3. **Case in which W_1 is a solid torus and W_2 a knot exterior.** Let $\widehat{N}$ denote a submanifold of $\mathbf{S}^3$ obtained from N by the Bonahon-Siebenmann re-embedding construction; this amounts to replacing W_2 by a solid torus, that we denote by U_2 (see Proposition 15). Recall that $\widehat{N}$ is diffeomorphic to N, and thus is given together with a Seifert foliation (indeed is a circle bundle).

Claim 5. — *The foliation on $\widehat{N}$ extends to $\mathbf{S}^3 = \widehat{N} \cup V_K^- \cup W_1 \cup U_2$.*

Let us admit the claim. In the solid torus W_1, the core must be an exceptional leaf, otherwise the annulus A would be boundary parallel through W_1. Thus, if we consider the original embedding $N \subset \mathbf{S}^3$, we see that K is a cable around the knot whose exterior is W_2.

Proof of Claim 5. — Since the complement of $\widehat{N}$ in $\mathbf{S}^3$ is a union of solid tori, the Seifert fioliation on $\widehat{N}$ extends either as a Seifert foliation or as a pseudo-foliation, say $\mathcal{F}$, on $\mathbf{S}^3$. By Corollary 25, it is enough to show that there cannot exist a pseudo-leaf in any of V_K^-, W_1, U_2.

Suppose that V_K^- would contain a pseudo-leaf. Then $\mathcal{F}$ would be the standard pseudo-foliation, with unique pseudo-leaf inside V_K^-. Hence the core of W_1 would be a regular leaf, and A would be boundary parallel through[2] the solid torus W_1. This would contradict the hypothesis on A, and is therefore impossible.

The same argument shows that U_2 does not contain a pseudo-leaf.

Suppose that W_1 contains a pseudo-leaf. Then the knot K is a regular leaf of the pseudo-foliation. But regular leaves are all isotopic in $\mathbf{S}^3 \smallsetminus U_2$, and they all bound discs. Hence K bounds a disc in $\mathbf{S}^3 \smallsetminus U_2$ that is a meridian disc in V_K. This disc lives also in the original situation, and this implies that the knot K is trivial, in contradiction with our hypothesis. □

5. Proof of Corollary 2 and of Theorem 3

Consider a 3-manifold M which satisfies the hypothesis of Theorem 3, and assume moreover that $\partial_1 M \approx \mathbf{T}^2$ is incompressible (see the comments which follow

[2]This is abusive, since W_1 stands here for a slightly expanded solid torus W_1^+ made up of W_1 and the appropriate part of N_A. But small lies can help the truth to be simpler.

Theorem 3). *We assume that $P \approx \pi_1(\partial_1 M)$ is not malnormal in $G = \pi_1(M)$, and we have to show that the JSJ piece V which contains $\partial_1 M$ is a Seifert manifold.*

By assumption, there exist $p_0, p_1 \in P \smallsetminus \{1\}$ and $g \in G \smallsetminus P$ such that $gp_0g^{-1} = p_1$. The elements p_0 and p_1 are represented by loops in $\partial_1 M$ which are freely homotopic in M. Hence there exists a (possibly singular) map $\varphi : \mathbf{A} \longrightarrow M$ of which the image connects these two loops; here, $\mathbf{A}$ is the standard annulus $\mathbf{S}^1 \times [0,1]$.

Let us check that φ is essential (compare with [32], Page 207). On the one hand, one component of $\partial\mathbf{A}$ generates $\pi_1(\mathbf{A}) \approx \mathbf{Z}$; its image by φ is $p_0 \neq 1$ (or $p_1 \neq 1$) so that, the group P being torsion-free, φ induces an injection of $\pi_1(\mathbf{A})$ into P, and therefore also into G. On the other hand, there is a spanning arc α in $\mathbf{A}$ that is mapped by φ to g; since $g \notin P$, the restriction $\varphi|\alpha$ is not homotopic relative to its boundary to an arc in ∂M.

From the Annulus Theorem 30, there exists an embedding $\psi : \mathbf{A} \longrightarrow M$ with $\psi(\partial\mathbf{A}) \subset \partial_1 M$. The annulus $\pi(\mathbf{A})$ is not boundary parallel (this is the meaning of ψ being essential in Theorem 30).

On the proof of Corollary 2. — For a non-trivial knot K, the argument of the few lines above show that, if the peripheral subgroup P_K is not malnormal in G_K, then there exists an annulus in E_K attached to T_K that is not boundary parallel. Hence Corollary 2 follows from Theorem 1. □

Remark. — The following is useless for our purpose but pleasant to know: the peripheral subgroup P_K of a knot group G_K is maximal abelian in G_K (a result of Noga, see Corollary 1 in [11]) and of infinite index in G_K (Theorem 10.6 in [16]).

We return to the situtation of Theorem 3. By Theorem 31, M has a family $\mathcal{T}$ of tori providing a JSJ decomposition in various pieces; recall that V denotes that piece which contains $\partial_1 M$.

Claim 6. — *There exists an embedded essential annulus in V, with at least one boundary component in $\partial_1 M$.*

Proof. — We know that there exists an embedded essential annulus ψ as above. Without loss of generality, we can assume that $\psi(\mathbf{A})$ is transversal to $\mathcal{T}$, so that the intersection $\psi(\mathbf{A}) \cap \mathcal{T}$ is a bunch $\mathcal{B}$ of circles. Let us agree that such a circle β is

- of the *first kind* if it bounds a disc in $\psi(\mathbf{A})$,
- of the *second kind* if it is boundary parallel in $\psi(\mathbf{A})$.

First, we get rid of circles of the first kind, by a classical argument. As a preliminary observation, note that a circle from $\mathcal{B}$ contained inside a circle of the first kind is also of the first kind. Thus, if there are circles of the first kind, one may choose one of them, say β, that is innermost, so that β bounds a disc Δ in $\psi(\mathbf{A})$ containing no element of $\mathcal{B}$ in its interior. Denote by T_i the torus of the family $\mathcal{T}$ that contains β; we have $\Delta \cap T_i = \partial\Delta$. If β was not contractible inside T_i, the disc Δ would be a compressing disc for T_i, and this is impossible since T_i is incompressible. Hence β bounds a disc, say δ, in T_i. The union $\delta \cup \Delta$ is a 2-sphere embedded in M; *since M is irreducible,* it bounds a 3-ball. We can first isotope Δ through this 3-ball and then push it slightly outside T_i to remove the intersection β.

Note that the irreducibility of M has played a crucial role above.

Iterating this operation, we can obtain eventually an annulus $\psi' : \mathbf{A} \longrightarrow M$ embedded in M without any circle of the first kind. Note that $\psi'(\mathbf{A})$ and $\psi(\mathbf{A})$ are isotopic in M, so that ψ' is also essential. The intersection $\psi'(\mathbf{A}) \cap \mathcal{T}$ is now a bunch $\mathcal{B}'$ of circles that are all of the second kind, namely that are boundary parallel in $\psi'(\mathbf{A})$.

The circles in $\mathcal{B}'$ decompose $\psi'(\mathbf{A})$ in a sequence of successive annuli. If the first of them is essential, keep it and stop. If it is inessential, then both its boundary components are in $\partial_1 M$, and we can repeat the same argument with the second annulus. After some time, we must encounter an essential annulus with one boundary component in $\partial_1 M$ and with empty intersection with $\mathcal{T}$, therefore an essential annulus entirely contained in V. □

Claim 7. — *With the notation of the previous claim, the manifold V is Seifert.*

Proof. — From the JSJ Decomposition Theorem, we know that V is Seifert or atoroidal. But, if V is atoroidal, it is also Seifert, thanks to the following proposition that we copy from Lemma 1.16 on Page 25 of [14]. □

Proposition 8. — *Let X be a compact, irreducible, and atoroidal 3-manifold. Assume that X contains an incompressible, ∂-incompressible annulus meeting only torus components of ∂M. Then X is a Seifert manifold.*

This ends the proof of the implication "P not malnormal $\Longrightarrow$ V Seifert" of Theorem 3.

Proof of the converse statement in Theorem 3. — We assume that V is a Seifert manifold. It is an irreducible manifold: one reason is noted as Comment (i) after Theorem 31, another one is that V has a boundary (Proposition 19).

We have the inclusions $\partial_1 M \subset V \subset M$ and the corresponding group homomorphisms $\mathbf{Z}^2 \approx \pi_1(\partial_1 M) \longrightarrow \pi_1(V) \longrightarrow \pi_1(M) = G$. As already noted in Section 1, we can assume that $\partial_1 M$ is incompressible in M, so that the inclusion $\pi_1(\partial_1 M) \longrightarrow G$ is an isomorphism onto the peripheral subgroup P of G; a fortiori, the inclusion $\pi_1(\partial_1 M) \longrightarrow \pi_1(V)$ is an injection. The homomorphism $\pi_1(V) \longrightarrow \pi_1(M)$ is also an injection (Remark (iii) after Theorem 31). Let us moreover remark that $\pi_1(\partial_1 M)$ is a proper subgroup of $\pi_1(V)$; otherwise, because of standard facts on fundamental groups of Seifert manifolds with boundaries (see Chapter 12 in [16]), V would be a thickened torus, and this has been ruled out. Summing up: $\pi_1(\partial_1 M) \approx \mathbf{Z}^2$ is a proper subgroup of $\pi_1(V)$.

Since V is a Seifert manifold, the torsion-free group $\pi_1(V)$ has an infinite cyclic normal subgroup, which is generated by the homotopy class of a regular fibre. By Proposition 2.viii of [15], it follows that $\pi_1(V)$ does not have any non-trivial malnormal subgroup. A fortiori, P is not malnormal in G. □

6. Corollary 2 as a consequence of Theorem 3

Corollary 2 is a consequence of Theorem 3 and of various facts on Seifert foliations (Appendix B) that are summed up in the following proposition.

Proposition 9. — *Let K be a non-trivial knot, E_K its exterior, T_K its boundary, and V the component of the JSJ decomposition of E_K containing T_K. If V is a Seifert manifold, then K is either a composite knot, or a torus knot, or a cable knot.*

Note. — One proof would be to show that, if V is Seifert, then there exists an essential embedded annulus in V. We will proceed differently, without using Theorem 1.

There are statements similar to our proposition in [19], Lemma VI.3.4, and [20], Lemma 14.8. The proof below is somewhat different, as it uses Seifert foliations and pseudo-foliations.

Proof of Proposition 9. — Denote by $T_1, \dots, T_r$ the connected components of ∂V distinct from T_K (possibly $r = 0$, if K a torus knot, since then E_K is Seifert foliated). For $j \in \{1, \dots, r\}$, let W_j denote the closure of the connected component of $\mathbf{S}^3 \smallsetminus T_j$ that does not contain V. If W_j was a solid torus, T_j would be compressible in E_K; but this would contradict the incompressibility of the JSJ tori in the decomposition of E_K. Hence, by Theorem 14, W_j is a knot exterior.

We consider the Bonahon-Siebenmann re-embedding $\widehat{V}$ of V in $\mathbf{S}^3$, summed up below in Proposition 15. This construction amounts to replace each W_j by a solid torus U_j. Then $\widehat{V}$ can be seen as the exterior E_L of a link $L := \widehat{K} \sqcup L_1 \sqcup \cdots \sqcup L_r$ with $r+1$ components.

Proposition 9 is now a consequence of the following claim. $\square$

Claim 10. — (i) *If the Seifert foliation on V does not extend to V_K (or, equivalently, if the Seifert foliation on $\widehat{V}$ does not extend to V_K), then $r \geqslant 2$ and K is the connected sum of r prime knots.*

(ii) *If the Seifert foliation on V extends to V_K, then $r \leqslant 1$. If $r = 0$, then K is a torus knot, and if $r = 1$ then K is a cable knot.*

Proof. — (i) By hypothesis, there exists a foliation on $\widehat{V}$ that does not extend to V_K. By Proposition 29, this foliation does extend as a pseudo-foliation on $\mathbf{S}^3$, with $\widehat{K}$ as a pseudo-leaf. By the uniqueness result for pseudo-foliations of $\mathbf{S}^3$, Corollary 25, the cores of the solid tori U_i are regular leaves. It follows from the description of composite knots à la Schubert (Subsection B.5) that K is a connected sum of r prime knots.

(ii) By hypothesis, there exists a foliation on V that extends to $M = V_K \cup V$; observe that the manifold M is irreducible (being Seifert and with boundary, Proposition 19). To show that $r \leqslant 1$, we proceed by contradiction and assume for some time that $r \geqslant 2$.

Consider the torus component T_1 of ∂M. Since M is irreducible and is not a solid torus, T_1 is incompressible in M (Lemma 13). Since T_1 bounds on the other side the cube with a knotted hole W_1, it is incompressible in W_1. Thus, van Kampen's theorem shows that $\pi_1(T_1)$ is a subgroup of $\pi_1(M \cup W_1)$. By a similar argument, $\pi_1(T_1)$ is still a subgroup of $\pi_1(M \cup W_1 \cup \cdots \cup W_r)$. But this is impossible since $M \cup W_1 \cup \cdots \cup W_r = \mathbf{S}^3$.

If $r = 0$, then K is a torus knot, since it is a leaf of a Seifert foliation of $\mathbf{S}^3$.

If $r = 1$, the same incompressibility argument as above shows that M is a solid torus (otherwise T_1 would be incompressible in $\mathbf{S}^3$). We know the classification of Seifert foliations on a solid torus: the space of leaves is a disc, and the number s of exceptional leaves is at most 1. If one had $s = 0$, the manifold V would be a thickened torus, and this is impossible because T_1 cannot be boundary parallel in a JSJ decomposition. Hence $s = 1$, and the knot K is not an exceptional leaf (this would again imply that V is a thickened torus). Hence K is a regular leaf of the Seifert foliation on the solid torus M, and this foliation has one exceptional leaf (the core of M). Thus K is a torus knot in M, not isotopic inside M to the core of M. It follows that this torus knot is satellised around the knot K_1 of which the exterior is W_1. This is exactly the cable situation. $\square$

Appendices on three-dimensional topology

Appendix A. Terminology and basic facts about 3-manifolds

This section is a reminder on some terminology for 3-manifolds, and classical results that we have used in Sections 4, 5 and 6 (Alexander, Dehn, Seifert, Waldhausen, Bing-Martin, Bonahon-Siebenmann). Recall the **standing assumption** agreed upon in Section 1:

all 3-manifolds and surfaces below are assumed to be compact, connected, orientable, and possibly with boundary,

but for a few exceptions which are either obvious or explicitely stated as such. Also, all maps are assumed to be smooth.

A map φ from a manifold N to a manifold M is **proper** if $\varphi^{-1}(\partial M) = \partial N$. A manifold S is **properly embedded** in a manifold M if it is embedded and if $\partial S = S \cap \partial M$.

A.1. **Irreducibility and parallelism.** A 3-manifold M is **irreducible** if any embedded 2-sphere bounds a 3-ball. For example, $\mathbf{S}^3$ is irreducible; indeed, it is a **theorem of Alexander** that any 2-sphere embedded in $\mathbf{S}^3$ bounds two 3-balls (see e.g. Theorem 1.1 in [14], Page 1). The only irreducible 3-manifold which has a 2-sphere in its boundary is the 3-ball. For the importance of the irreducibility hypothesis, see for example the proof of Claim 6.

Let M be a manifold of dimension m. Let S_0, S_1 be two manifolds of the same dimension $n < m$, with S_0 properly embedded in M and S_1 either properly embedded in M or embedded in ∂M. Then S_0 and S_1 are **parallel** if there exists an embedding of a thickened manifold

$$\psi \,:\, S \times [0,1] \longrightarrow M$$

such that

(i) $\psi(S \times \{0\}) = S_0$ and $\psi(S \times \{1\}) = S_1$,
(ii) $\psi(\partial S \times [0,1]) \subset \partial M$.

If $\partial M \neq \emptyset$, a manifold S_0 properly embedded in M is **boundary parallel**, or **∂-parallel**, if there exists a manifold S_1 embedded in ∂M such that S_0 and S_1 are parallel.

Consider for example the case with $M = A$ an annulus and S of dimension 1. There are three isotopy classes of properly embedded arcs in an annulus A: one class with the two ends of the arc in one component of ∂A, one class with the two ends of the arc in the other component of ∂A, these are boundary parallel, and the class of the so-called **spanning arcs** with one end in each component of ∂A, equivalently with $A \smallsetminus \{\text{arc}\}$ connected.

Recall that a simple closed curve in a surface is **essential** if it is not homotopic to a point, equivalently if it does not bound an embedded disc. A circle embedded in A is essential if and only if it is boundary parallel, and is then a **core** of A. Note that a core of A and a spanning arc of A, appropriately oriented, have intersection number $+1$.

A.2. **Incompressible and ∂-incompressible surfaces.** Let S be a surface properly embedded in a 3-manifold M and γ a simple closed curve in the interior of S. A **compressing disc** for γ is a disc D embedded in M such that $\gamma = \partial D = D \cap S$. The surface S is **incompressible** if, for any simple closed curve γ in the interior of S which has a compressing disc D, there exist a disc D' in S such that $\partial D' = \gamma$ (equivalently: γ is null-homotopic in S). Note that our definition is different from that of [16] for properly embedded surfaces which are discs or spheres; for us, these are **always** incompressible.

Mutatis mutandis, this definition of "incompressible" applies to boundary components of M.

A non-connected surface (for example ∂M in some situations) is incompressible if each of its connected components is so (see e.g. Section 1.2 in [14]).

A connected surface S properly embedded in M, or a component of ∂M, is incompressible if and only if the induced homomorphism of groups $\pi_1(S) \longrightarrow \pi_1(M)$ is injective. This follows from **Dehn's Lemma and the loop theorem**; see Corollary 3.3 in [14]. (It is important here that S is two-sided, but this follows from our *standing assumption,* according to which both S and M are orientable.)

For example, the boundary T_K of a non-trivial knot exterior E_K is incompressible, and the boundary of a handlebody of genus $g \geqslant 1$ is compressible.

Given a (not necessarily connected) surface S properly embedded in a 3-manifold M, the manifold M_S^* obtained from M by **splitting** M **along** S is the complement in M of a regular open neighbourhood of S (observe that S is two-sided, being orientable in an orientable manifold). We quote now Theorem 1.8 in [34].

Proposition 11 (Waldhausen). — *Let M and S be as above; assume that S is incompressible. Then the connected components of M_S^* are irreducible if and only if M is irreducible.*

Let S be a surface with boundary $\partial S \neq \emptyset$ properly embedded in a 3-manifold M with boundary $\partial M \neq \emptyset$. For an arc α properly embedded in S, a **compressing disc** is a disc D embedded in M with:

- $\alpha = D \cap S$,
- $\beta := D \cap \partial M$ is an arc in ∂D,
- $\partial D = \alpha \cup \beta$ and $\partial\alpha = \partial\beta = \{\text{two points in } \partial D\}$.

(Observe that such a D is never properly embedded in M since the interior of α is disjoint from ∂M.) The surface S is **∂-incompressible** if, for any arc α properly embedded in S with D as above, α is boundary parallel in S.

Proposition 12. — *Let M be an irreducible 3-manifold. Assume that the boundary of M has some torus compoments; let $\partial_T M$ denote the union of these.*

Let S be a surface properly embedded and incompressible in M, with $\emptyset \neq \partial S \subset \partial_T M$. Then either S is ∂-incompressible or S is a boundary parallel annulus.

In particular, if ∂M is a union of tori, an annulus properly embedded and incompressible in M is either ∂-incompressible or boundary parallel.

Proof. — We refer to Lemma 1.10 of [14]. □

Remark. — If $\partial_T M$ is compressible, it follows from Lemma 13 below that M is a solid torus. It is known that, in a solid torus, an incompressible surface which is not a disc is necessarily an annulus parallel to the boundary (Lemma 2.3 in [34]).

A.3. **Complements of tori in the 3-sphere.** Let T be a torus embedded in $\mathbf{S}^3$. By Poincaré-Alexander duality, the complement $\mathbf{S}^3 \smallsetminus T$ has two connected components, and their closures U_1, U_2 have T as a common boundary. By the theorem of Alexander recalled in Subsection A.1, the manifolds U_1 and U_2 are irreducible.

The following Theorem 14 is also due to Alexander (see [14], Page 11). The proof below (unlike that of Alexander!) uses Dehn's Lemma. Our preliminary Lemma 13 is well-known to specialists.

Lemma 13. — *Let M be an irreducible 3-manifold; assume that the boundary ∂M has a component $\partial_1 M$ which is a compressible torus.*

Then M is a solid torus; in particular, ∂M is connected.

Proof. — Let D be a compressing disc for $\partial_1 M$ and let E be a small open tubular neighbourhood of D. Let $M_D^* = M \smallsetminus E$ be the result of splitting M along D. By construction, the boundary ∂M_D^* contains a 2-sphere, consisting of "most of" $\partial_1 M \cup \partial E$. By the irreducibility assumption, this 2-sphere (viewed now in M) bounds a 3-ball B in M. Then $V \doteqdot B \cup E$ is a solid torus, because it is orientable and obtained by attaching E along ∂B as a 1-handle.

This solid torus is closed in M since it is compact. It is also open by Brouwer's theorem of invariance of domain (see the remark below). It follows that $V = M$. □

Remark (On Brouwer's Theorem). — The following result is (a restatement of what is) found in books, see e.g. [9, Proposition 7.4]: *an injective continuous mapping $g : N_1 \longrightarrow N_2$ between two manifolds N_1, N_2, of the same dimension and without boundary, is open.* Let now M_1, M_2 be manifolds of the same dimension, with boundary, and let $\partial' M_2$ be the union of some of the connected components of ∂M_2. *If an injective continuous mapping $f : M_1 \longrightarrow M_2$ is such that $f(\partial M_1) = \partial' M_2$, then f is open.* This is a straightforward consequence of the previous statement, applied to the natural map g induced by f, with domain the double $N_1 = M_1 \cup_{\partial M_1} M_1$ of M_1 and target the interior of the double $N_2 = M_2 \cup_{\partial' M_2} M_2$.

The **core** of a solid torus U embedded in a 3-manifold M is $h(\mathbf{S}^1 \times \{0\})$, where U is the image of an embedding $h : \mathbf{S}^1 \times \mathbf{D}^2 \longrightarrow M$ of the standard solid torus (the core is well-defined up to isotopy).

Theorem 14 (Alexander). — *Let T be a torus embedded in $\mathbf{S}^3$ and let U_1, U_2 be the closures of the connected components of $\mathbf{S}^3 \smallsetminus T$.*

At least one of them, say U_1, is a solid torus, say with core C_1, so that U_2 is the exterior of the core C_1. The curve C_1 is unknotted in $\mathbf{S}^3$ if and only if U_2 is also a solid torus.

Proof. — If T were incompressible in both U_1 and U_2, the group $\pi_1(T)$ would inject in $\pi_1(U_1)$ and $\pi_1(U_2)$, by Dehn's lemma. Since $\mathbf{S}^3 = U_1 \cup_T U_2$, it would also inject in the amalgamated sum

$$\pi_1(\mathbf{S}^3) \;=\; \pi_1(U_1) *_{\pi_1(T)} \pi_1(U_2),$$

by the Seifert–van Kampen theorem, and this is absurd. Upon exchanging U_1 and U_2, we can therefore assume that T is compressible in U_1. Lemma 13 implies that U_1 is a solid torus.

If U_2 is also a solid torus, then T is unknotted, by definition. □

Alexander's theorem is strongly used in the following construction, that we propose to call the **Bonahon-Siebenmann's re-embedding construction**. In [4, beginning of § 2.2], this is called a *splitting.* On page 326 of [6] and with the notation of our Proposition 15, the embedding of $\widehat{Z}$ in $\mathbf{S}^3$ is called the *untwisted re-embedding.*

Let Z be a 3-manifold embedded in $\mathbf{S}^3$, with boundary a non-empty disjoint union of tori $\partial Z = T_1 \sqcup \ldots \sqcup T_r$. For $j \in \{1, \ldots, r\}$, denote by W_j the closure of the connected component of $\mathbf{S}^3 \smallsetminus T_j$ which does not contain Z. Assume that the notation is such that $W_1, \ldots, W_\ell$ are knot exteriors and $W_{\ell+1}, \ldots, W_r$ solid tori, for some ℓ with $0 \leqslant \ell \leqslant r$. The purpose of the construction is to obtain a situation with $\ell = 0$, namely with Z re-embedded as the exterior of an appropriate link in the 3-sphere.

For $j \in \{1, \ldots, \ell\}$, denote by μ_j a meridian and by λ_j a parallel of T_j viewed as the boundary of the solid torus $\overline{\mathbf{S}^3 \smallsetminus W_j}$; orient these so that they become a basis of $H_1(T_j, \mathbf{Z})$. Let U_j denote a solid torus, with boundary endowed with an oriented meridian μ'_j and an oriented parallel λ'_j. Define inductively a sequence of manifolds $M_0 = \mathbf{S}^3, M_1, \ldots, M_\ell$; for $j \in \{1, \ldots, \ell\}$, the manifold M_j is obtained by gluing U_j to the closure of $M_{j-1} \smallsetminus W_j$, in such a way that μ_j is glued to λ'_j and λ_j to μ'_j. Observe that the construction provides an embedding of Z in M_j, and that the components of the complement of the image of Z in M_j can be naturally identified with $U_1, \ldots, U_j, W_{j+1}, \ldots, W_r$. Since M_j has a Heegaard decomposition of genus one[3] and has the homology of the 3-sphere, M_j is diffeomorphic to $\mathbf{S}^3$. In particular, M_ℓ is diffeomorphic to $\mathbf{S}^3$, and we denote it by $\mathbf{S}^3$ again; we denote by $\widehat{Z}$ the image of the embedding of Z in this "new" $\mathbf{S}^3$.

For reference, we state the result of this construction as:

Proposition 15. — *Let Z be a 3-manifold embedded in $\mathbf{S}^3$, with boundary a non-empty disjoint union of tori.*

There exists a submanifold $\widehat{Z}$ of $\mathbf{S}^3$ that is diffeomorphic to Z and that is the exterior of a link in $\mathbf{S}^3$.

[3] A manifold N that has a Heegaard decomposition of genus one is diffeomorphic to either the 3-sphere, or $\mathbf{S}^1 \times \mathbf{S}^2$, or a lens space. Indeed, the median torus of such a decomposition is the boundary of two solid tori, and has therefore two meridians μ, μ'. If $\delta = \Delta(\mu, \mu')$, with the notation of Subsection 2.1, then $\pi_1(N) \approx \mathbf{Z}/\delta\mathbf{Z}$, and N is $\mathbf{S}^1 \times \mathbf{S}^2$, or $\mathbf{S}^3$, or a lens space, if the value of δ is 0, or 1, or $\geqslant 2$. See Chapter 2 of [16].

Using automorphisms of the solid tori we could show that, given two results $\widehat{Z}, \widehat{Z}' \subset \mathbf{S}^3$ of the construction, there exists a diffeomorphism h of $\mathbf{S}^3$ such that $h(\widehat{Z}) = \widehat{Z}'$.

The following result is due to Bing and Martin. A manifold with boundary homeomorphic to the exterior of a non-trivial knot in $\mathbf{S}^3$ is picturesquely called in [1] a **cube with a knotted hole**; we also use **knot exterior**. We insist that a knot exterior is the exterior of a **non-trivial** knot; observe that Bing-Martin's result does not carry over to the exterior of a trivial knot.

Proposition 16 (Bing-Martin). — *Consider a solid torus U in $\mathbf{S}^3$ and a knot exterior W contained in the interior of U.*

Then there exists a 3-ball B in the interior of U such that $W \subset \overset{\circ}{B}$.

Proof. — Since W is the exterior of a non-trivial knot, the inclusion of ∂W in W induces an injection of $\pi_1(\partial W) \approx \mathbf{Z}^2$ into $\pi_1(W)$, by Dehn's lemma (the proof of Bing and Martin *does not* use Dehn's lemma). Since

$$\mathbf{Z} \approx \pi_1(U) \approx \pi_1(U \smallsetminus \overset{\circ}{W}) *_{\pi_1(\partial W)} \pi_1(W),$$

by the Seifert–van Kampen theorem, this implies that $\pi_1(\partial W)$ does not inject in $\pi_1(U \smallsetminus \overset{\circ}{W})$. Hence, by Dehn's lemma, there exists a compressing disc D for ∂W in $U \smallsetminus \overset{\circ}{W}$. The union of W and of a thickening of this disc D is the 3-ball we are looking for. □

Appendix B. Seifert foliations and pseudo-foliations

B.1. **Seifert foliations.** In this paper, a foliation always means a **foliation by circles of a 3-manifold** M, namely a partition of M in circles with the usual regularity hypothesis [13]. A foliation is **oriented** if all its leaves are coherently oriented.

For example, fixed-point free actions of the rotation group $SO(2)$ on 3-manifold provide oriented foliations (see Proposition 17).

Standard actions of $SO(2)$ on the solid torus provide important examples called **standard foliated solid tori**. More precisely (we follow Page 299 of [25]), parametrise the solid torus $\mathbf{S}^1 \times \mathbf{D}^2$ by $(e^{i\psi}, \rho e^{i\theta})$, with $0 \leqslant \rho \leqslant 1$ and $0 \leqslant \psi, \theta < 2\pi$. Given two coprime integers μ, ν with $0 \leqslant \nu \leqslant \mu$, the corresponding **standard linear action** of $SO(2) = \{z \in \mathbf{C} \mid |z| = 1\}$ on the solid torus is defined by

$$SO(2) \times \mathbf{S}^1 \times \mathbf{D}^2 \longrightarrow \mathbf{S}^1 \times \mathbf{D}^2, \qquad (z, e^{i\psi}, \rho e^{i\theta}) \longmapsto (z^\mu e^{i\psi}, z^\nu \rho e^{i\theta}).$$

This action is always effective[4]. It is free if and only if $\mu = 1$ (this implies $\nu = 0$ or $\nu = 1$); in this case, the orbits are fibers of a product fibration $\mathbf{S}^1 \times \mathbf{D}^2 \longrightarrow \mathbf{D}^2$. If $\mu > 1$ (this implies $1 \leqslant \nu \leqslant \mu - 1$), the action is free on the complement of the core of equation $\rho = 0$; this core is the **exceptional orbit**, and the other orbits are the *regular* ones.

For $\mu = 0$ and $\nu = 1$, the same formula defines an action with regular orbits in meridian discs, and with the core as fixed point set. See Subsection B.2 below.

A **Seifert foliation** is a circle foliation such that each leaf has a neighbourhood that is a union of leaves and that is isomorphic to a standard foliated solid torus. Each neighbourhood isomorphic to a standard foliated solid torus with $\mu \geqslant 2$ contains an **exceptional leaf**, and all other leaves are **regular leaves**. Many authors use "Seifert fibration" for our "Seifert foliation", but our terminology is motivated by the fact that these in general **are not** circle bundles, and because "fibration" is already used in many other situations. A **Seifert manifold** is a manifod which admits a Seifert foliation.

[4] Recall that an action of a group G on a set X is **effective** (or *faithful*) if, for any $g \in G$, $g \neq e$, there exists $x \in X$ with $gx \neq x$.

Let M be a 3-manifold given with a Seifert foliation. Let $\mathcal{B}$ be the corresponding **space of leaves**, viewed here as a surface, possibly with boundary, possibly non-orientable (we *do not* view $\mathcal{B}$ as an orbifold), and let $p : M \longrightarrow \mathcal{B}$ denote the quotient map. Since M is orientable, $\mathcal{B}$ is orientable if and only if the Seifert foliation is orientable. We mark the points in $\mathcal{B}$ which correspond to exceptional leaves in M and we denote by $\check{\mathcal{B}}$ the surface obtained from $\mathcal{B}$ by removing disjoint small discs around the marked points. It is standard that $\mathcal{B}$, with appropriate decorations, provides a complete description of M and its Seifert foliation (see e.g. Theorem 2.4 in [3]).

The following proposition is not deep, but useful. It shows an equivalence between Seifert foliations and leaves on the one hand, and fixed point free $SO(2)$-actions and orbits on the other hand.

PROPOSITION 17. — *The leaves of an oriented* **Seifert** *foliation on a* 3*-manifold* M *are the orbits of a fixed point free action of* $SO(2)$ *on* M*, and conversely.*

Proof. — Any fixed point free $SO(2)$-action gives rise to an oriented Seifert foliation; this is an immediate consequence of the existence of a slice for the action (see [23], or the middle of Page 304 in [25]).

For the converse implication, we can quote [26] or [25], who prove that any Seifert data can be realised by a $SO(2)$-action. Alternatively, see [10], Page 80, for a more direct approach. □

The next theorem, from [10], is much deeper than Proposition 17. It is remarkable enough to be stated here, even if we do not use it elsewhere in this paper. (Recall that our 3-manifolds are compact, connected, and orientable.)

THEOREM 18 (Epstein). — *The leaves of an oriented* circle *foliation on a* 3*-manifold* M *are the orbits of a fixed point free action of* $SO(2)$ *on* M*.*

The following result is due to Waldhausen. For the first claim, see e.g. Lemma VI.7 in [17] or Proposition 1.12 in [14]. For the second claim, see Corollary 3.2 [29].

PROPOSITION 19. — *A Seifert manifold is either irreducible, or* $\mathbf{S}^1 \times \mathbf{S}^2$*, or the connected sum of two projective spaces.*

In particular, a Seifert manifold with boundary is irreducible.

Hence, if M *is a Seifert manifold with boundary, either* ∂M *is incompressible or* M *is a solid torus.*

Finally, let us quote a proposition which shows that there are plenty of incompressible annuli and tori in Seifert manifolds. For the proof, see Page 127 in [19].

PROPOSITION 20. — *Let* M *be a Seifert manifold, and let* $p : M \longrightarrow \mathcal{B}$ *denote the projection on its space of leaves. Let* α *be a properly embedded arc in* $\mathcal{B}$ *which avoids the marked points; set* $A = p^{-1}(\alpha)$*. Let* γ *be a simple closed curve in* $\mathcal{B}$ *which avoids the marked points and which is orientation-preserving; set* $T = p^{-1}(\gamma)$*. Then :*

(i) A *is a properly embedded annulus in* M *which is incompressible.*
(ii) A *is boundary parallel in* M *if and only if* α *is boundary parallel in* $\mathcal{B}$*.*
(iii) T *is a properly embedded torus in* M*; it is compressible in* M *if and only if* γ *bounds a disc in* $\mathcal{B}$ *which contains at most one marked point.*

These are examples of so-called **vertical** annuli and tori in Seifert manifolds. For a precise description of annuli and tori in 3-manifolds, see Theorems 3.9 and 3.5 in [3], respectively.

B.2. **Seifert pseudo-foliations, general facts.** Consider the standard solid torus $\mathbf{U} := \mathbf{S}^1 \times \mathbf{D}^2$ and its core $C = \{0\} \times \mathbf{S}^1$. The **standard Seifert pseudo-foliation** of $\mathbf{U}$ is the partition $\mathcal{F}_0$ of $\mathbf{U}$ in the points of the circle C and the circles $\gamma_\rho \times \{z\}$, where γ_ρ is a circle of radius $\rho > 0$ in $\mathbf{D}^2$ centred around the origin and z is a point of $\mathbf{S}^1$; we call C the **pseudo-leaf** and the other circles the **leaves** of $\mathcal{F}_0$.

More generally, let us define a **Seifert pseudo-foliation** of a 3-manifold M to be a partition of M in circles and points, which restricts to a Seifert foliation outside a finite disjoint union of solid tori $V_1, \dots, V_r$, and to standard Seifert pseudo-foliations inside these solid tori. Pseudoleaves and leaves of such a pseudo-foliation are defined naturally; leaves can be either regular or exceptional, as in Seifert foliations. We insist that we assume $r \geqslant 1$; in other words:

it is part of the definition of a Seifert pseudo-foliation that it contains at least one pseudo-leaf.

By definition, a Seifert pseudo-foliation of M as above restricts to a Seifert foliation on $M \smallsetminus (\overset{\circ}{V_1} \sqcup \cdots \sqcup \overset{\circ}{V_r})$. From now on, we will write **pseudo-foliation** instead of Seifert pseudo-foliation.

For example, and as a consequence of the classification of circle foliations on the 2-torus, every circle foliation $\mathcal{F}_\partial$ on the boundary of a solid torus extends to a Seifert foliation or a pseudo-foliation $\mathcal{F}$ on the solid torus itself. More precisely, $\mathcal{F}$ is a pseudo-foliation if the leaves of $\mathcal{F}_\partial$ are meridians, and $\mathcal{F}$ is a Seifert foliation in all the other cases. Hence, if V is a submanifold of a closed manifold W (e.g. of $\mathbf{S}^3$) with $W \smallsetminus V$ a union of solid torus, any Seifert foliation or pseudo-foliation on V extends to a foliation or a pseudo-foliation on W.

A Seifert pseudo-foliation is **oriented** if its restriction to the complement of the pseudo-leaves is oriented.

Proposition 17 has an analogue for pseudo-foliations:

Proposition 21. — *The leaves of an oriented pseudo-foliation on a* 3*-manifold* M *are the orbits of an effective action of* $SO(2)$ *on* M *with fixed points, and conversely.*

Proof. — That any effective action of $SO(2)$ with fixed points gives rise to a pseudo-foliation is again a straightforward consequence of the slice theorem (see the proof of Proposition 17). For the converse, we believe that Epstein's proof carries over. Alternatively, see Theorem 1 in [25] and Corollary 2b, with $t = 0$, in [26]. □

The **space of orbits** $\mathcal{B}$ of a pseudo-foliation of a manifold M is again a surface (possibly with boundary, possibly non orientable); we use also "space of leaves" instead of "space of orbits", even though this is abusive, since the restriction of the projection $M \longrightarrow \mathcal{B}$ to each pseudo-leaf is a bijection. There are again marked points in $\mathcal{B}$, corresponding to exceptional leaves in M and we define $\check{\mathcal{B}}$ as above; also, it is again true (as in B.1) that $\mathcal{B}$ together with appropriate decorations provides a complete description of M and its pseudo-foliation. Here are two basic examples.

(i) For the standard pseudo-foliation of the solid torus $\mathbf{U}$, the space of orbits is an annulus. One component of its boundary is the space of orbits of the 2-torus $\partial\mathbf{U}$, the other component corresponds to the pseudo-leaf, which is the core of $\mathbf{U}$, and there are no marked points. The canonical projection $p : \mathbf{U} \longrightarrow \mathcal{B}$ restricts to a bijection from the core of $\mathbf{U}$ onto one component of the boundary of the annulus $\mathcal{B}$.

Conversely, let M be a manifold with a pseudo-foliation such that the corresponding space $\mathcal{B}$ is an annulus with one boundary component being the space of orbits of ∂M, and without marked points; then M is a solid torus with the standard pseudo-foliation.

(ii) On $\mathbf{S}^3 = \mathbf{R}^3 \cup \{\infty\}$, the standard action of $SO(2)$ by rotations around an axis defines a pseudo-foliation with one pseudo-leaf. The space of orbits $\mathcal{B}$ is the 2-disc $\mathbf{D}^2$, and the boundary of the disc represents the fixed points of the action. Conversely, if a pseudo-foliation on M gives rise to such a 2-disc, then M is the 3-sphere, the leaves are the circular orbits of the standard action of $SO(2)$, and the pseudo-leaf is the circle of fixed points.

To simplify the discussion, we assume from now on that M has no boundary. In particular, boundary components of $\mathcal{B}$ are in bijection with pseudo-leaves of M.

B.3. **Seifert pseudo-foliations on closed 3-manifolds.** To emphasise the difference between foliations and pseudo-foliations, we state as a lemma an observation due to Waldhausen (see Page 90 of [34], the discussion of Condition 6.2.4).

Lemma 22 (Waldhausen). — *Let $\mathcal{F}$ be a pseudo-foliation on a closed 3-manifold M, let $p : M \longrightarrow \mathcal{B}$ be the projection on the space of orbits, and let β be an arc properly embedded in $\mathcal{B}$ which avoids the marked points.*

Then $p^{-1}(\beta)$ is a 2-sphere embedded in M.

Proof. — Remove a little interval at each extremity of β. Let β^* denote the closure of the complement of these intervals. Then $p^{-1}(\beta^*)$ is an annulus. The inverse image of a little interval is a disc, indeed a meridian disc in the tubular neighbourhood of the pseudo-leaf. Thus $p^{-1}(\beta)$ is an annulus with a disc glued on each of its boundary components; hence $p^{-1}(\beta)$ is a 2-sphere. □

Easy and standard topological considerations show that Lemma 22 has the following consequences.

Proposition 23. — *Let the notation be as in the previous lemma.*

(i) *The sphere $p^{-1}(\beta)$ separates M if and only if β separates $\mathcal{B}$, thus producing a connected sum decomposition of M.*
(ii) *If β does not separate $\mathcal{B}$, then the sphere $p^{-1}(\beta)$ produces a factor $\mathbf{S}^1 \times \mathbf{S}^2$ in M.*
(iii) *If $\mathcal{B}$ is a disc with one marked point, then M is a lens space.*
(iv) *The sphere $p^{-1}(\beta)$ bounds a 3-ball if and only if β is boundary parallel in $\check{\mathcal{B}}$.*

Comments. — Claim (i) is straightforward. Claim (ii) holds by classical arguments (Lemma 3.8, Page 27, in [16]). For (iii), see Page 301 of [25]. Claim (iv) is a consequence of the previous ones. □

The following result can either be easily deduced from the considerations above, or recovered as a special case of a result of Orlik and Raymond, written in terms of $SO(2)$-actions with fixed points (Page 299 of [25]).

Proposition 24. — *Let M be a closed orientable manifold that admits an orientable pseudo-foliation. Then M is either a 3-sphere, or $\mathbf{S}^1 \times \mathbf{S}^2$, or a lens space, or a connected sum of these.*

Corollary 25. — *Let M be a homology 3-sphere; assume that M is furnished with a pseudo-foliation. Then $M = \mathbf{S}^3$ is the standard sphere, and the pseudo-foliation, up to diffeomorphism, is given by the standard action of the rotation group $SO(2)$ around an axis of the sphere $\mathbf{S}^3 = \mathbf{R}^3 \cup \{\infty\}$, as in Example (ii) of the end of B.2. In particular, the pseudo-foliation has exactly one pseudo-leaf.*

Proof. — Let $\mathcal{F}$ be a pseudo-foliation on M; by our next Lemma 26, $\mathcal{F}$ is orientable.

If the space of leaves $\mathcal{B}$ of $\mathcal{F}$ either had at least two boundary components or had genus $\geqslant 1$, there would exist in $\mathcal{B}$ an arc β as in Lemma 22, and Proposition 23.ii would imply that $\mathbf{Z} = H_1(\mathbf{S}^1 \times \mathbf{S}^2, \mathbf{Z})$ is a direct factor of $H_1(M, \mathbf{Z})$; this is impossible because M is a homology sphere. Hence $\mathcal{B}$ is a disc.

If this disc had just one marked point, M would be a lens space by Proposition 23.iii, and this is again impossible since M is a homology sphere. If this disc had two or more marked points, M would be a connected sum of lens spaces, and this is equally impossible. Hence $\mathcal{B}$ is a disc without marked points.

This implies that $M = \mathbf{S}^3$, and that $SO(2)$ acts as stated in the corollary. □

A Seifert foliation (or pseudo-foliation!) on the 3-sphere is necessarily orientable. More generally:

Lemma 26. — *Let Z be a 3-manifold embedded in a homology 3-sphere M. If Z has a Seifert foliation with space of leaves $\mathcal{B}$, then $\mathcal{B}$ is orientable.*

Proof. — Let us first recall that, in a homology 3-sphere M, any embedded surface S without boundary is orientable. Indeed, using homology and cohomology with coefficients the field $\mathbf{F}_2$ with two elements (so that $H^2(S) \approx \mathbf{F}_2$ for any closed surface S, orientable or not), Alexander duality

$$H^2(S) \approx H_0(M \smallsetminus S)/H_0(\text{point}) \approx \mathbf{F}_2$$

shows that S is two-sided, and therefore orientable.

To prove the lemma, it is enough to show that any simple closed curve γ in $\mathcal{B}$ which avoids the marked points is two-sided. Given such a γ, consider the surface $S := p^{-1}(\gamma)$. Since S is orientable in an orientable manifold, S is two-sided; it follows that γ is two-sided.

[Note that, since S is orientable and is a circle bundle over a circle, S is a torus]. □

We end this subsection by translating Propositions 19 and 24 in terms of $SO(2)$-actions:

Corollary 27. — *Let M be a closed orientable 3-manifold.*

(i) *If M admits a fixed-point free $SO(2)$-action, then M is either irreducible, or $\mathbf{S}^1 \times \mathbf{S}^2$, or the connected sum of two projective spaces.*
(ii) *If M admits a non-trivial $SO(2)$-action with fixed points, then M is $\mathbf{S}^3$, or $\mathbf{S}^1 \times \mathbf{S}^2$, or a lens space, or a connected sum of these.*
(iii) *If M admits $SO(2)$-actions of the two kinds, with fixed points and without, then M is $\mathbf{S}^3$, or $\mathbf{S}^1 \times \mathbf{S}^2$, or a lens space.*

Thus, the list of $SO(2)$-actions with fixed points if far more restricted than the list of actions without fixed points.

It is a natural temptation to hope for a general theory which would encompass Seifert foliations and pseudo-foliations; but we should not give in this, as Waldhausen has warned us (see the *Bemerkung* on Page 91 of [34]). Indeed, Seifert manifolds are irreducible (up to a small number of exceptions, see Proposition 19); irreducibility is a crucial ingredient of their theory and classification. On the contrary, "most" pseudo-foliated manifolds are reducible (see Proposition 24); as a consequence, this "general theory" would be worthless.

B.4. **Seifert manifolds embedded in $\mathbf{S}^3$.** We begin by stating a standard characterization of torus knots, used above in Subsection 4.2.

Proposition 28 (Seifert foliations on knot exteriors). — *A knot K such that E_K carries a Seifert foliation is a torus knot or the trivial knot.*

Proof. — There are two cases to distinguish.

(i) Suppose that the foliation extends to V_K, providing a Seifert foliation of $\mathbf{S}^3$. By Seifert's classification of the Seifert foliations on the 3-sphere [30], K is either a torus knot or the trivial knot.
(ii) Suppose that the foliation, say $\mathcal{F}$, does not extend to V_K. Then the induced foliation on T_K is necessarily a foliation by meridians, so that $\mathcal{F}$ extends to a pseudo-foliation $\mathcal{F}'$ on $\mathbf{S}^3$. By Corollary 25, $\mathcal{F}'$ has a unique pseudo-leaf, which is K and which is a trivial knot. □

Proposition 28 suggests to distinguish two types of links, as follows (it is a result from [7]).

Proposition 29 (Seifert foliations on link exteriors). — *Let $L = L_1 \sqcup \cdots \sqcup L_r$ be a link in $\mathbf{S}^3$, with $r \geqslant 2$ components. Assume that E_L admits a Seifert foliation $\mathcal{F}$.*

(i) *If $\mathcal{F}$ extends to a foliation of the 3-sphere, then L is obtained by selecting r leaves of a Seifert foliation of $\mathbf{S}^3$. Links of this type are called* ***torus links****.*
(ii) *If $\mathcal{F}$ does not extend as a foliation, then it extends to a pseudo-foliation of the 3-sphere, necessarily with a unique pseudo-leaf which is a component of L, say L_1. The other components $L_2, \ldots, L_r$ of L are meridians around the pseudo-leaf L_1.*

Proof. — This is a straightforward consequence of Corollary 25. ☐

Let us finally define terms which enter our Corollary 4. A link L is **unsplittable** if, for every 2-sphere S in $\mathbf{S}^3$ disjoint from L, all components of the link L are in the same connected component of $\mathbf{S}^3 \smallsetminus S$. A **torus sublink** is a part $L_{i_1} \sqcup \cdots \sqcup L_{i_s}$ of a link $L_1 \sqcup \cdots \sqcup L_r$, for some subsequence of indices with $1 \leqslant i_1 < \cdots < i_s \leqslant r$; it is itself a torus link. A **connected sum operation** on two links $L = L_1 \sqcup \cdots \sqcup L_r$ and $L' = L'_1 \sqcup \cdots \sqcup L'_s$ consists in connecting *just one* component L_j of L with one component L'_k of L'.

B.5. **Composite knots à la Schubert.** We revisit Schubert's description of composite knots [28] in terms of pseudo-foliations and the re-embedding construction. Consider a solid torus $\mathbf{U}$ embedded in an unknotted way in $\mathbf{S}^3$, an integer $r \geqslant 2$, and r disjoint meridian discs $D_1, \ldots, D_r$ in $\mathbf{U}$; thicken these discs a little bit; the thickened discs separate $\mathbf{U}$ in r closed 3-balls $B_1, \ldots, B_r$.

Let K be a knot embedded in the interior of $\mathbf{U}$. We assume that K intersects each disc D_i transversely in exactly one point. Hence K runs across each thickened disc in a little unknotted arc, and $A_i := K \cap B_i$ is a properly embedded arc in B_i, for $i = 1, \ldots, r$. We assume moreover that the arc A_i is knotted in B_i. Denote by K_i the knot obtained from A_i by adding an unknotted arc outside B_i. Then, by construction-definition, the knot K is the connected sum $K_1 \sharp \cdots \sharp K_r$ of the knots $K_1, \ldots, K_r$.

Denote by V_K a thin tubular neighbourhood of K in $\mathbf{U}$, and set $T_K = \partial V_K$. Consider a little collar of T_K inside V_K, denote by T'_K the component of its boundary which is inside V_K, and by V_K^- the smaller tubular neighbourhood of K with boundary T'_K.

For $i \in \{1, \ldots, r\}$, let W_i be the closure of $B_i \smallsetminus (V_K \cap B_i)$. Thus W_i is a cube with a knotted hole; this hole is indeed knotted, since A_i is knotted by hypothesis. Note that W_i is the exterior of the knot K_i defined above. Set $T_i := \partial W_i$, which is a 2-torus.

On the one hand, define

$$\Sigma := \mathbf{S}^3 \smallsetminus \big(V_K^- \sqcup W_1 \sqcup \ldots \sqcup W_r\big)$$

(the $\sqcup$ indicate disjoint unions), and observe that $\partial\Sigma = T'_K \sqcup T_1 \sqcup \cdots \sqcup T_r$. On the other hand, consider the link $L = L_0 \sqcup L_1 \sqcup \cdots \sqcup L_r$ in $\mathbf{S}^3$ obtained from the standard pseudo-foliation by selecting the pseudo-leaf L_0 and m regular leaves $L_1, \ldots, L_r$. The Bonahon-Siebenmann re-embedding $\widehat{\Sigma}$ of Σ in $\mathbf{S}^3$ is obtained by replacing each cube with a knotted hole W_i by a cube with an **unknotted hole**, namely by a solid torus, say U_i. Thus $\widehat{\Sigma}$ is the exterior of the link L defined just above.

From that description, we see that the link exterior $\widehat{\Sigma}$ is diffeomorphic to a product $F \times \mathbf{S}^1$, where F is a planar surface with $r + 1$ boundary components $\partial_0 F, \ldots, \partial_r F$. The product $\partial_i F \times \mathbf{S}^1$ is the boundary of a little tubular neighbourhood of L_i. Hence Σ is also Seifert foliated and diffeomorphic to $F \times \mathbf{S}^1$. Since the foliation on $\widehat{\Sigma}$ does not extend to a tubular neigbourhood of $L_0 = \widehat{K}$, the foliation Σ does not extend to this neighbourhood of K. In both foliations (of Σ and of $\widehat{\Sigma}$), the leaves on the boundary of a tubular neighbourhood are meridians.

To obtain the composite knot K from the link L, we have just to replace each cube with an unknotted hole U_i by W_i. After this replacement, the link component $L_0 = \widehat{K}$ is changed into K.

Remark (On the necessity of the condition $r \geqslant 2$). — In the connected sum point of view, we wish the sum to be non-trivial, namely involving at least two non-trivial factors. In the JSJ point of view, if the Seifert foliation on V does not extend to V_K and if $r = 1$, then the JSJ torus T_1 is boundary parallel (parallel to T_K), and this contradicts the JSJ conditions.

Appendix C. The annulus theorem and the JSJ decomposition

C.1. **The annulus theorem.** A first major ingredient of our proof of Theorem 3 is the annulus theorem, announced together with the torus theorem by Waldhausen [36]; a detailed proof of the annulus theorem was given in [8]. Before stating the theorem, we recall some terminology.

We denote by $\mathbf{A}$ the standard annulus $\mathbf{S}^1 \times [0,1]$, and by $\partial_0\mathbf{A} = \mathbf{S}^1 \times \{0\}$, $\partial_1\mathbf{A} = \mathbf{S}^1 \times \{1\}$ the two components of its boundary $\partial\mathbf{A}$. Recall that *spanning arcs* in $\mathbf{A}$ have been defined in A.1. A proper map from the standard annulus to a 3-manifold M, say $\varphi : (\mathbf{A}, \partial\mathbf{A}) \longrightarrow (M, \partial M)$, is **essential** if the induced homomorphism $\pi_1(\mathbf{A}) \longrightarrow \pi_1(M)$ is injective and if, for a spanning arc α in $\mathbf{A}$, the restriction $\varphi|\alpha$ is not homotopic rel its boundary to an arc in ∂M.

Theorem 30 (Annulus Theorem). — *Let M be a compact orientable 3-manifold and let $\varphi : \mathbf{A} \longrightarrow M$ be an essential map from the annulus to M. Then there exists an essential embedding $\psi : \mathbf{A} \longrightarrow M$ such that, for $i = 0$ and $i = 1$, the image $\psi(\partial_i\mathbf{A})$ lies in the same connected component of ∂M as $\varphi(\partial_i\mathbf{A})$.*

On the proof. — The last part of our formulation, from “such that”, is not explicit in Theorem 3 of [8], but it follows from their proof. □

C.2. **The JSJ decomposition.** The second major ingredient in our proof of Theorem 3 is the **JSJ decomposition of 3-manifolds**, as stated below. Traditionally, JSJ refers explicitely to Jaco-Shalen-Johannson, and also implicitely to Waldhausen [36].

A 3-manifold M is **atoroidal** (other authors use “**simple**”) if any incompressible torus in M is boundary parallel (as defined in A.1). Recall that $M^*_{\mathcal{T}}$ denotes the manifold obtained by splitting M along $\mathcal{T}$.

Theorem 31 (JSJ Decomposition). — *Let M be an irreducible 3-manifold.*

In the interior of M, there exists a family $\mathcal{T} = \{T_1, \dots, T_r\}$ of disjoint tori that are incompressible and not parallel to components of ∂M, with the following properties:

(i) *each component of $M^*_{\mathcal{T}}$ is either a Seifert manifold or atoroidal;*
(ii) *the family $\mathcal{T}$ is minimal among those that have Property (i).*

Moreover, such a family $\mathcal{T}$ is unique up to ambient isotopy.

Proof. — The reference we like best is [3, Theorem 3.4]. See also (among others) Theorem 1.9 in [14], Page 169 of [19], and the comments in the much shorter [18].

We insist that, in these references, M is allowed to have a boundary. See e.g. [14, Page 1] and, more implicitely, [19, Page 1]. □

The connected components of $M^*_{\mathcal{T}}$ are the **pieces of the JSJ-decomposition**, and $\mathcal{T}$ is the **characteristic torus family**. Observe that, by Condition (ii), no piece can be a thickened torus, unless M itself is a thickened torus (in which case $\mathcal{T}$ is the empty family, and M has a unique component, itself).

Remarks (Comments on the statement of Theorem 31). —

(i) By Proposition 11, each piece is irreducible.

(ii) A 3-manifold can be both atoroidal and Seifert; examples include the solid torus, the thickened torus, exteriors of torus knots, and some manifolds without boundary. The complete list, which is rather short, can be found in [19] (Page 129, and Lemma IV.2.G). Thus, the "or" in (i) of Theorem 31 is not exclusive.

(iii) Let V be a piece. Since the T_j's are incompressible, the inclusion $V \subset M$ induces an injection of $\pi_1(V)$ into $\pi_1(M)$. This follows from an appropriate version of the Seifert–van Kampen theorem.

C.3. **Beyond Theorems 30, 31, and 3.** The Cannon-Feustel theorem 30 can be thought of as the original annulus theorem. There exists a stronger result, due to Johannson (the "enclosing theorem" of [20]) and Jaco-Shalen (the "mapping theorem" of [19], see their remark in the middle of Page 56); see also Page 173 of [17]. This requires more general JSJ-like decompositions, with a characteristic surface $\mathcal{A}$ (rather than $\mathcal{T}$) composed of *annuli and tori*, and with pieces which can be *atoroidal, or Seifert manifolds or* $\mathbf{I}$*-bundles* ($\mathbf{I}$ is the unit interval). The existence and unicity of such an $\mathcal{A}$ is closely related to the strong result just alluded to, and to a general homotopy annulus theorem for essential mapping from a Seifert manifold to a 3-manifold. For a statement which is both precise and readable, we refer to Theorem 3.8 in [3]. Clearly, such strong theorems imply easily our Theorem 3.

To give some idea of the homotopy annulus theorem, let us state it in the particular case of a manifold with boundary a union of tori, a case for which no annuli are needed in the JSJ-like decomposition (see [24], Page 38). Note that Theorem 32 *is not used* in this paper.

Homotopy Annulus Theorem 32 (particular case). — *Consider the situation of Theorem 31, and assume* moreover *that ∂M is a non-empty union of incompressible tori. Assume that there exists an essential map (not necessariliy an embedding) $\varphi : (\mathbf{A}, \partial\mathbf{A}) \longrightarrow (M, \partial M)$ of the annulus into M.*

Then there exists a homotopy

$$\varphi_t \;:\; (\mathbf{A}, \partial\mathbf{A}) \longrightarrow (M, \partial M), \qquad 0 \leqslant t \leqslant 1,$$

such that $\varphi_0 = \varphi$ and φ_1 is an essential map (not necessarily an embedding) into a Seifert piece of the JSJ decomposition (in particular, this decomposition has at least one Seifert piece).

Moreover, in that Seifert piece, there exist embedded essential annuli (vertical ones).

We cannot expect that φ_1 is homotopic to an embedding. But, in a Seifert component, there are plenty of incompressible vertical annuli. Hence we can strenghten the conclusion of the annulus theorem, and conclude to the existence of an essential embedding with vertical images inside some Seifert component.

Appendix D. On the terminology and the literature

D.1. **On essential annuli.** Embedded annuli and tori play the key role in our arguments. Since the literature concerning the related terminology is in our opinion rather messy, we review the following definitions.

Let M be a bounded 3-manifold and A an annulus. A mapping

$$f : (A, \partial A) \longrightarrow (M, \partial M)$$

is **W-essential**, or essential in the sense of Page 24 of [37], if the induced morphism of groups $\pi_1(A) \longrightarrow \pi_1(M)$ and the induced morphism of pointed sets $\pi_1(A, \partial A) \longrightarrow \pi_1(M, \partial M)$ are both injective (f need not be an embedding). Observe that $\pi_1(A, \partial A)$ has precisely two elements, the base point and the non-trivial element represented by a spanning arc; it follows that this definition of "essential" is equivalent to that of [32] (Page 206) or that of [8] (Page 220).

A mapping $f : (A, \partial A) \longrightarrow (M, \partial M)$ is **non-degenerate** if the homomorphism $\pi_1(A) \longrightarrow \pi_1(M)$ is injective and if f is not homotopic (as a map of pairs) to a map $g : (A, \partial A) \longrightarrow (M, \partial M)$ with $g(A) \subset \partial M$ (we follow [19], Pages 121–122). If M is irreducible and if ∂M is incompressible, Lemma IV.1.3 of [19] shows that f is non-degenerate in this sense if and only if f is W-essential.

It seems that the terminology with "essential" becomes standard; see for example [3], just before his Theorem 2.14.

Thus, for an annulus properly embedded and incompressible in an irreducible manifold M with ∂M a union of tori, we have a priori several notions:

(i) it can be **∂-incompressible**, or equivalently **not boundary parallel** (see Proposition 12),

(ii) it can be **W-essential**, or equivalently **non-degenerate**.

In fact, **these four notions are equivalent**.

Indeed, on the one hand, "boundary parallel" clearly implies "degenerate". On the other hand, Lemma 5.3 of [35] contains more than is necessary to show that "degenerate" implies "boundary parallel". Here is a weakened version of this Lemma 5.3, with Waldhausen's notation.

Lemma 33. — *Let M be an irreducible 3-manifold. Let G be an incompressible boundary component of M, and let F be an incompressible surface properly embedded in M such that $\partial F \subset G$. Suppose that there exists a homotopy $H : F \times \mathbf{I} \longrightarrow M$ such that $H(F \times \{0\}) = F$ and $H(\partial(F \times \mathbf{I}) \smallsetminus (F \times \{0\})) \subset G$.*

Then F is boundary parallel, and more precisely is parallel to a surface contained in G.

D.2. **On the terminology of Raymond and Orlik.** For the reader who wishes to read [26] and [25], we offer the following dictionnary.

- M is a compact and connected 3-manifold on which $SO(2)$ acts. For these authors, M can be non-orientable; *but we assume in this paper that M is orientable*, so that their symbol ϵ takes always the value o (small "o").
- M^* is the orbit space (our $\mathcal{B}$); in our case, M^* is a compact orientable surface, and $g \geqslant 0$ is its genus.
- F is the fixed point set and F^* is its homeomorphic image in M^*; the number of connected components of F is denoted by h in [26] and by $\hbar$ in [25].
- E denotes the set of exceptional orbits; its cardinal can be any non-negative integer. SE is the set of special exceptional orbits; since M is orientable here, $SE = \emptyset$, so that its cardinal t is always equal to 0.

Hence, in our case, $M_{\epsilon,s,\hbar,t}$ is always $M_{o,g,\hbar,0}$.

We do not need to comment on the Seifert invariants, namely on b, which is a variant of the Euler class (caution: there is a sign problem there), and on (α_j, β_j), which are the usual Seifert invariants.

The projective plane is denoted by P, the Klein bottle by K. The "non-orientable handle" N, which is the non-trivial $\mathbf{S}^2$-bundle over $\mathbf{S}^1$, does not play any role for us.

D.3. **Concerning hyperbolic geometry.** Part of the Thurston revolution consists in recognizing the importance of hyperbolic geometry in the subject of 3-manifolds. For example:

Theorem 34. — *Let M be an irreducible 3-manifold, and $\mathcal{T}$ a characteristic torus family, as in Theorem 31. Suppose moreover that $\partial M \neq \emptyset$.*

Then each connected component of $M^*_{\mathcal{T}}$ is a Seifert manifold or a hyperbolic manifold.

In other words, each atoroidal component of $M^*_{\mathcal{T}}$ is Seifert or hyperbolic.

Theorem 34 is a consequence of the hyperbolization theorem for Haken manifolds, that was announced by Thurston during lectures at Princeton during the Spring 1977, and later precised in [33] (see in particular Theorem 2.5), where Thurston initiated a program that was completed near the year 2000; see [2]. Note that Theorem 34 does not cover all closed manifolds, for which the hyperbolization theorem is due to Perelman.

Hyperbolization theorems can be used to give quick proofs of some of the results of Section 1. Consider for example a non-trivial knot K. Assume that K is neither a satellite knot nor a torus knot. Then E_K has a hyperbolic structure [33, Corollary 2.5], and it follows that the peripheral subgroup P_K is malnormal in the group G_K of K [15, Example 6].

But other cases would be less straightforward. Consider for example a Whitehead double, say K; it is a satellite knot and is not a cable knot. By Corollary 2, P_K is malnormal in G_K. But the JSJ-decomposition of E_K has more than one component, E_K does not have a geometric structure, so that the few lines above are not sufficient for a proof of Corollary 2.

We leave it to other readers or writers to discuss how the results of Section 1 follow from what we know now on the geometrization conjecture and on relatively hyperbolic groups.

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Pierre DE LA HARPE & Claude WEBER
Section de mathématiques, Université de Genève, C.P. 64, CH–1211 Genève 4, Suisse
Pierre.delaHarpe@unige.ch
Claude.Weber@unige.ch