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THREE-DIMENSIONAL INTERACTION OF SHOCKS IN IRROTATIONAL FLOWS

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À la mémoire de Michelle, de laquelle j'ai beaucoup appris.

The general *d*-dimensional Riemann problem raises naturally the question of resolving the interaction of *d* planar shocks merging at a point. In gas dynamics, we may consider only standing shocks. This problem has received a satisfactory answer in dimension $d = 2$ (see [3, 4]). We investigate the 3-dimensional case. We restrict to the irrotational case, in order to keep the complexity of the solution within reasonable bounds. We show that a new kind of waves appears downstream, which we call a *conical wave*. When the equation of state is that of Chaplygin/von Kármán, we give a complete mathematical answer to this problem. This involves the existence and uniqueness of a complete minimal surface in a hyperbolic space, with prescribed asymptotics.

Keywords: Shock waves; elliptic PDE; characteristic boundary-value problem.

AMS Subject Classification: 35L65, 35L67, 35J25, 35J70, 76N10

1. From the Riemann Problem to Conical Waves

In the theory of conservation laws, the Riemann problem plays a prominent role, for at least four reasons. First, the assumption that the solution is self-similar reduces the complexity of the Cauchy problem, giving hope to establish more accurate results, either qualitative or quantitative. Second, the Riemann problem is central in numerical analysis because it is a building block of difference schemes of the Godunov family. Third, the Riemann problem is a benchmark for the comparison of the performances of numerical codes. Finally, it plays a central role in Glimm's existence proof for initial data with small total variation in one space variable.

A system of conservation laws is a system of PDEs of the form

$$\partial_t U + \operatorname{div}_x F(U) = 0, \quad t > 0, \ x \in \mathbb{R}^d. \tag{1.1}$$

The unknown $U(t, x)$ belongs to some convex set $\mathcal{U}$ of $\mathbb{R}^n$. We always assume that this system is hyperbolic: the symbol

$$A(U; \xi) := \sum_{j=1}^{d} \xi_j \nabla_U F^j$$

is diagonalizable with real eigenvalues. When $|\xi| = 1$, these eigenvalues are velocities of propagation in the direction ξ of infinitesimal disturbances. We make the more precise assumption of *constant hyperbolicity*: the number m of distinct eigenvalues of $A(\xi; U)$ is independent of U and $\xi \neq 0$. We have $m \leq n$, and the eigenvalues are smooth functions of U and $\xi \neq 0$. If $m = n$, one speaks of *strict* hyperbolicity. If $m < n$, some eigenvalue(s) has multiplicity $\nu \geq 2$ and a theorem of Boillat tells us that the corresponding characteristic field is linearly degenerate in Lax's terminology.

A paradigm of conservation laws is the Euler system for gas dynamics, in which the state is described by a mass density $\rho > 0$, an entropy s and a velocity field u. The pressure p is a prescribed smooth function of (ρ, s). We assume that $\partial p / \partial \rho$ is positive for the relevant values of (ρ, s), its square root $c(\rho, s)$ being the sound speed. The PDEs are the conservation of mass, momentum and energy

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.2}$$

$$\partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \tag{1.3}$$

$$\partial_t E + \operatorname{div}((E + p)u) = 0, \tag{1.4}$$

where the conserved quantities U_j $(1 \leq j \leq d + 2)$ are the mass density, the linear momentum ρu and the total energy E. The latter splits into its kinetic and internal parts

$$E := \frac{1}{2} \rho |u|^2 + \rho e.$$

1.1. *The Riemann problem*

In one space dimension $(d = 1)$, the Riemann problem is the Cauchy problem between two constant states U_ℓ (for $x < 0$) and U_r (for $x > 0$). It has been well understood since the seminal paper of Lax [7]. Roughly speaking, the Riemann problem produces m waves separating $m+1$ constant states $U_0 = U_\ell, U_1, \ldots, U_m = U_r$. For data with small oscillations about a generic state $\bar{U}$, each wave is either a rarefaction, a shock or a contact discontinuity; these are called *simple waves* (see the figure on p. 545).

The situation is more intricate in dimension $d = 2$. In the simplest Riemann problem, the plane at $t = 0$ splits into three sectors around the origin, the initial data being constant in each of them. Each line separating two sectors forms a 1-dimensional Riemann problem in the normal direction, and the complete solution is expected to coincide with these 1-dimensional patterns away from the influence
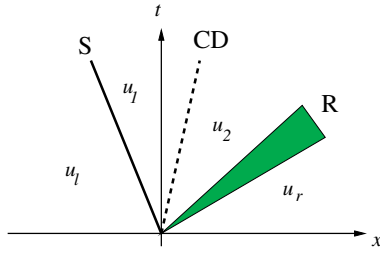
domain of the origin. This latter domain is a cone $\mathcal{C}$ with tip at the origin, whose section at $t = 1$ is compact. The solution is genuinely 2-dimensional in $\mathcal{C}$, which means that the rank of $\nabla_x U$ equals 2, see Fig. 1.

There is no available complete strategy for solving the 2D Riemann problem. The first step is of course the resolution of the 1D problems mentioned above. They produce simple waves. The second step consists in solving the pairwise interaction between the latter waves coming from different directions and traveling toward each other; for instance in Fig. 1, the shocks U_2/U_{23} and U_2/U_{12} interact. This is what we call a primary interaction. It may be simple or complicated. *Simple* means that the incoming waves are straight discontinuities; then the interaction corresponds to a kind of 1D Riemann problem, though in a spatial direction, instead a temporal one. On the contrary, *complicated* means that at least one incoming wave is a rarefaction fan; then the interaction is genuinely 2D. In order to be able to solve a Riemann problem completely, we will restrict our attention in this paper to situations where all the primary waves are straight discontinuities. This is guaranteed if the system (1.1) has all its characteristic fields *linearly degenerate*. In this favorable case, each 1D Riemann problem produces straight discontinuities. Then the second step may be continued, by solving pairwise interactions of transmitted waves, *ad libitum*.
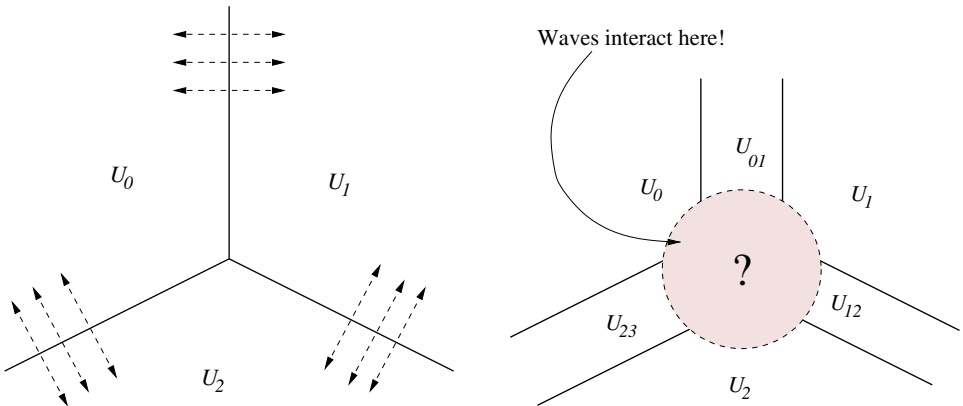

Fig. 1.   Left, the piecewise constant data for a 2D Riemann problem. The initial discontinuities form 1D Riemann problems. Right, partial view of the solution. The 1D Riemann problems have been solved, here with $m = 2$. They remains to match the intermediate states in the central region.

Unfortunately (or fortunately, if we like new mathematical problems), resolving pairwise interactions is not enough to obtain the whole solution of a 2D Riemann problem. When $m \geq 2$, the type of the system governing self-similar solutions is not given *a priori*; it depends upon the solution itself. There is a zone where this type is elliptic (or mixed hyperbolic–elliptic if $m \geq 3$). The second step can be continued as long as one stays in the hyperbolic region, but it comes to a dead end after finitely many pairwise interactions. The remaining part of the solution obeys some kind of elliptic Boundary-value problem (BVP) and has to be determined globally. In the linear case this elliptic zone is the influence cone of the origin, which can be identified directly from the symbol of the differential operator. However, many relevant systems, like gas dynamics, are not linear and this domain is not known *a priori*: the BVP is actually a FBVP, a free boundary-value problem. As we shall see below, this rule suffers one exception: if the characteristic fields are linearly degenerate, then the boundary of the elliptic zone is characteristic and can be identified explicitly at the end of the second step. This happens in gas dynamics when the equation of state is that of Chaplygin; see Appendix A.1 for this notion.

1.1.1. *Pairwise interaction of planar discontinuities*

We assume $d = 2$, but this paragraph applies also to $d = 3$ whenever the data does not depend on one coordinate (planar waves). Suppose that the initial data equals a constant state U_0 in a sector S, and that it equals other constant states $U_\pm$ in the sectors next to S. Each pair $(U_0, U_\pm)$ yields a 1D Riemann problem, in which the backward wave $W^\pm$ separates U_0 from a state $U_{1\pm}$. The planar waves $W^\pm$ travel toward U_0 and are transversal to each other. They delimit a sector $S + t\bar{v}$, in which the solution of the full 2D Riemann problem still equals U_0; this constancy is a consequence of the finite velocity of wave propagation. The waves $W^\pm$ interact along the line L of equation $x = t\bar{v}$. Because of the finite wave velocity, we expect that there be a conical neighborhood $\mathcal{K}$ of L in which the solution is determined by the waves $W^\pm$ only, and does not depend at all upon the rest of the data. As mentioned before, we suppose that $W^\pm$ are shocks or contact discontinuities. Then up to the choice of a moving frame in which L is vertical (that is, $\bar{v} = 0$), the source of the interaction is constant in sectors and we thus expect that U is both steady and self-similar in $\mathcal{K}$,

$$U(t, x) = \bar{U}\left(\frac{x}{|x|}\right).$$

Hereabove, $\bar{U}$ equals $U_{1\pm}$ in sectors on both sides of S. We have to solve a kind of 1D Riemann problem in the variable x only, in a direction $\mathbf{e}$ opposite to S, for the steady system

$$\mathrm{div} F(U) = 0, \tag{1.5}$$

with "initial data" $U_{1\pm}$. Notice that for data of moderate strength, the steady system is hyperbolic in the direction $\mathbf{e}$, because of the Lax shock inequality applied

to the backward waves $W^\pm$. Following Lax's analysis, we expect that the interaction is described by $m + 1$ constant states $U_0 = U_{1-}, U_1, \ldots, U_m = U_{1+}$, separated by m simple waves.

An easy example is that of a linear system with $m = 2$ (say, acoustics). The incident discontinuities do not really interact, as they cross each other without changing direction (principle of superposition). The (constant) state behind the interaction point is $U \equiv U_{1-} + U_{1+} - U_0$. In nonlinear problems, U will also be piecewise constant, as long as the transmitted waves are discontinuities (contact or shock waves).

Quasi-linear behavior. In a pairwise interaction of planar waves, the solution often resembles that of the linear case in the sense that the solution is piecewise constant with discontinuities occurring along straight lines. The discrepancy between the linear and nonlinear cases is that in general the transmitted states are not linear functions of $(U_0, U_\pm)$. For instance if $m = 2$, the transmitted state is not equal to $U_{1+} + U_{1-} - U_0$. Also, the directions of the outgoing waves differ from those of the incoming waves $W^\pm$. However, the distortion is small when $U_{1\pm}$ are close enough to U_0.

Increasing versus stable complexity. In a pairwise interaction, both incoming waves disappear, while m outgoing waves are created. The number of waves is thus multiplied by a factor $m/2$. Let us examine the case where $m \geq 3$. Because there are several stages of pairwise interactions, the picture can become extremely complicated as one approaches the elliptic zone. Even worse, the system (1.5) is not just elliptic in the latter zone, but mixed hyperbolic–elliptic. On the contrary, if $m = 2$ the number of waves does not increase as we resolve an interaction. If the first step produced 6 waves, two for each of the three 1D Riemann problems, the number of waves remains equal to 6 as we go forward. We thus expect that the elliptic zone be delimited by 6 arcs, each one separating the genuinely 2D pattern from a constant state. In addition, the type of (1.5) in the central zone is elliptic. This is definitely the simplest nontrivial situation. For this reason, the only 2D Riemann problems to have been solved in the existing literature have a wave number $m = 2$, see [1, 9, 2, 8] and the references herein. We shall adopt the same restriction. In gas dynamics, this amounts to assuming that the flow is barotropic and irrotational.

Pairwise interaction of planar shocks in gas dynamics is extensively studied in the book of Courant and Friedrichs [1]. We recall some basic calculations in Appendix A.2.

1.2. 3*D* shocks: *Triple interaction*

In a 3D Riemann problem, the domain $\mathbb{R}^3$ is split into conical cells in which the initial data is constant. The boundary of a cell is piecewise planar; in other words, the section of a cell is polygonal. As in the previous paragraph, we assume $m = 2$, in order to avoid a relevant, but dissuasive, complexity.

The simplest situation is that of cells of triangular sections. Let us select such a cell K, in which the data is a constant U_0. Denote the neighbor states by $\bar{U}_1, \bar{U}_2, \bar{U}_3$.
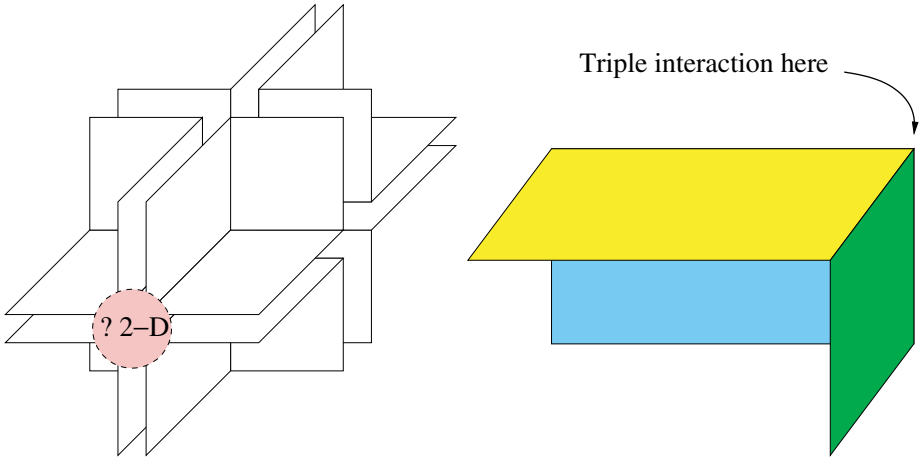
Fig. 2.   Left, a 3D Riemann problem with eight constant states in triangular cells (here cells are octants). Each of the 12 1D Riemann problems are resolved first. Four of them (front) contribute to a 2D Riemann problem. Around the origin (not visible on the figure, the pattern is genuinely 3D. Right, the front-bottom-left detail is the data of our triple interaction. The three planes surround the conical cell K.

Each pair $(U_0, \bar{U}_j)$ is separated by a planar sector, which forms a 1D Riemann problem. It yields a backward planar wave W_j, which separates U_0 from another constant state U_j. At time $t > 0$, the W_j's surround a pyramidal cone $K + t\bar{v}$ in which $U \equiv U_0$; its tip moves at constant speed $\bar{v}$. Up to the choice of a moving frame, we may assume that the shocks W_j are steady: $\bar{v} = 0$. See Fig. 2.

In the sequel, we adopt the convention that whenever the indices i, j, k are present simultaneously, we have $\{i, j, k\} = \{1, 2, 3\}$.

Suppose now that each of the W_j's are either shocks or contacts, so that the pattern made of $U_0, \ldots, U_3$ is piecewise constant. Any two among the three waves (say W_i and W_j) yield a 2D interaction along the edge ℓ_k of K, because the pattern made of U_0, U_i, U_j is constant in the direction of ℓ_k. Such an interaction has been discussed in the previous paragraph; it produces in general a new constant state denoted U^k, behind two transmitted planar waves. When the data is close to a constant, the interaction is approximately linear and therefore $U^k \sim U_i + U_j - U_0$. The states U^k are therefore pairwise distinct, and we need at least secondary interactions, plus presumably a genuinely 2D pattern. It is this *triple* interaction that we wish to analyze here.

Because $\bar{v} = 0$, we are looking for a solution that is both self-similar in spacetime and stationary, thus satisfies

$$U = U\left(\frac{x}{|x|}\right).$$

We emphasize that U is *not* the solution U_{RP} of the complete Riemann problem. It is only its restriction to a conical domain, where U_{RP} depends only upon $U_0, \ldots, U_3$

and not upon the remaining data. Even if the data consists only on four states and sectors, U_{RP} depends upon the whole $(U_0, \bar{U}_1, \bar{U}_2, \bar{U}_3)$. Focusing on a small part of the full Riemann problem is legitimate because of the property of finite velocity of wave propagation.

We make the generic assumption that the fastest backward wave[a] in the Riemann problem between U_0 and $\bar{U}_j$ is nontrivial; it is W_j. Finally, we assume that in the pairwise interaction of planar shocks or contact discontinuities, the outgoing waves are themselves shocks or contact discontinuities. This is obviously true if the system is linearly degenerate, because every planar simple wave is a contact discontinuity.

When the restricted data (U_1, U_2, U_3) oscillates moderately around U_0, the stationary system (1.5) is hyperbolic in a direction $\mathbf{e}$ pointing outward of the cone $(-\mathbf{e} \in K)$; this is true because the W_j's are backward waves (again the Lax shock inequalities). The situation resembles much a 2D Riemann problem, if we think of the plane $\mathbf{e} \cdot x = 0$ as that of "initial time". Then the Cauchy data is constant in sectors, taking the values $U_1, U^3, U_2, U^1, U_3, U^2$ in this circular order. Resolving the triple interaction is therefore solving this Cauchy problem. We point out that this sort of 2D Riemann problem is not a generic one, because each discontinuity (U_i, U^j) is coherent, in the sense that it solves its own Riemann problem. Remember that these are precisely the transmitted waves obtained after pairwise interactions between the discontinuities U_0/U_i and U_0/U_k.

1.2.1. *The interplay of nonlinearity and dispersion*

At first glance, this coherence seems to simplify the interaction. This is the case at least for a linear system: thanks to linear superposition, the complete solution U is piecewise constant and takes eight values

$$U_0, U_1, \, U_2, \, U_3$$

$$U^1 = U_2 + U_3 - U_0, \quad U^2 = U_3 + U_1 - U_0, \quad U^3 = U_1 + U_2 - U_0$$

$$U^0 = U_1 + U_2 + U_3 - 2U_0$$

in octants separated by three planes, each plane being the extension of the support of an incoming wave W_j. On the contrary, the solution of a linear 2D Riemann problem with a non-coherent data would be piecewise constant away from a conical neighborhood $\mathcal{C}$ of $\mathbf{e}$, and genuinely 2-dimensional in $\mathcal{C}$, see [6].

For nonlinear problems such as gas dynamics, an important question is therefore whether the coherence of the incoming waves is dominant as in the linear case (and thus the triple interaction is more or less trivial, say piecewise constant), or if the nonlinearity (even a small amount of it) shakes this nice picture. We shall see in Sec. 2.2 that nonlinearity does play a role, even in a case where the system is linearly degenerate (gas with a Chaplygin equation of state).

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[a]In gas dynamics, it is the backward pressure wave.

Actually, nonlinearity is not the only property at work, because a nonlinear pairwise interaction often leads to a piecewise constant solution, two incoming shocks producing two outgoing shocks. Dispersion plays a role too. Dispersion is the fact that the wave velocities depend nonlinearly on the frequency. In gas dynamics, pressure waves are dispersive because the wave velocity $u \cdot \xi + c(\rho, s)|\xi|$ is not a linear function of ξ; on the contrary, the entropy and vorticity waves travel at the non-dispersive velocity $u \cdot \xi$. Of course, dispersion is not sufficient to produce genuinely 2D patterns; we have seen above that nonlinearity is also necessary. It is thus a combination of both nonlinearity and dispersion that gives rise to genuinely 2D patterns.

When the triple interaction of coherent shocks or contact discontinuities is not piecewise constant, we call the genuine 2D regime a *conical wave*. This terminology has been employed for flows past sharp obstacles as soon as in [3], but the kind of waves we are interested in is something else. We have not encountered it in the literature so far.

1.3. *Main results*

We shall prove in this paper the following statement.

Theorem 1.1. *Consider an isentropic irrotational flow for a gas obeying the Chaplygin equation of state. Then, for moderate initial strength, the triple interaction of steady shocks admits a unique solution made of primary and secondary interactions, plus a smooth conical wave.*

The Chaplygin equation of state and its main features are described in Appendix A.1. Let us recall that the Euler system of gas dynamics with this equation of state is closely related to differential geometry and to the theory of strings and branes. In one space dimension, there is only one equivalence class 2×2 system with linearly degenerate (but not linear) fields. This class contains the system governing an isentropic Chaplygin flow, as well as the Born–Infeld model of electrodynamics.

The proof of Theorem 1.1 uses explicit resolutions of pairwise interactions of planar waves, plus the analysis of a degenerate elliptic boundary value problem governing the potential of the velocity. As observed by Lihe Wang,[b] a suitable transformation of the unknown makes this BVP equivalent to that governing a complete minimal surface of equation $x_3 = f(x_1, x_2)$ in the space $H := \mathbf{S}^2 \times (0, \infty)$ endowed with the Riemannian metric

$$ds^2 = \frac{1}{\sinh x_3^2}(d\sigma^2 + dx_3^2). \tag{1.6}$$

Hereabove $d\sigma$ is the standard metric on the 2-sphere. This Riemannian space is complete, its infinity being $(\mathbf{S}^2 \times \{0\}) \cup \{\infty\}$. Our boundary condition $f = 0$ on a
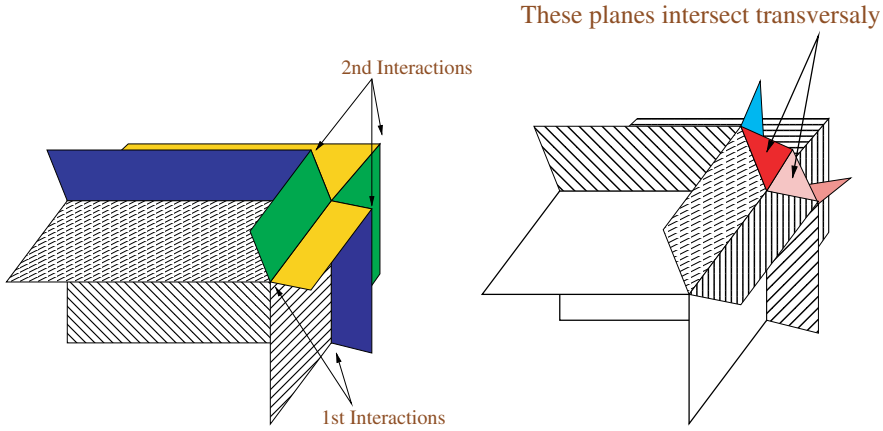
[b]Personal communication.

Fig. 3.   Left, shocks from the data interact first along the edges ℓ_k. The outgoing shocks W_i^k meet and will in turn interact. Right, these secondary interactions have been resolved. They produce six planar shocks. Because of the nonlinearity, these shocks are pairwise distinct (there would be only three distinct ones in linear acoustics). This is the cause of the conical wave.

given strictly convex closed curve can be viewed as an asymptotics. This problem is similar to that encountered in our previous study of the 2D Riemann problem [9], where the key result was equivalent to the existence and uniqueness of a complete minimal surface in the 3D Poincaré half-space $\mathbb{H}_3$ with given asymptotic convex curve. The metric was

$$ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2),$$

whose curvature is constant negative. In the present situation, the scalar curvature is still negative, but not constant.

1.3.1. *Solving the triple interaction*

Secondary interactions. Even if the linear picture does not remain correct in the presence of nonlinearity, it is a valuable guide. We first have primary interactions between waves W_i and W_j along the line ℓ_k. We denote W_i^k the transmitted waves separating the states U_i and U^k.

The outgoing waves W_i^k and W_i^j, which bound the domain occupied by the state U_i, approach each other and eventually interact along a line L^i, producing two other planar waves W^{ki}, W^{ji} and a new state $\hat{U}^i$ behind. This is what we call the *secondary* interactions.

A general rule, based on the unique continuation principle for hyperbolic Cauchy problem, is that a planar shock wave separating two constant states V_1 and V_2 can be continued until

- either it meets another wave,
- or it meets one of the characteristic cones associated with $V_{1,2}$.

This is the principle that allows us to use the primary and secondary interactions in the analysis of the triple interaction. If the states $U_0, \ldots, U_3$ are not too far away from each other, the characteristic cones are close to that of U_0; the latter is approximately the inner tangent cone to the octant $-K$. Thus the wave W_j cannot meet the characteristic cones associated with U_0 and U_j before it intersects W_i and W_k. The same phenomenon happens for the waves W_i^j.

A free-boundary problem. Let us now have a look beyond the secondary interactions. In the linear case, the waves W^{ik} are tangent to $\mathcal{C}(U^i)$ and $\mathcal{C}(\hat{U}^k)$, because they are characteristic. When passing to the nonlinear case, the picture depends on whether the corresponding characteristic field is linearly degenerate or genuinely nonlinear:

- If it is linearly degenerate (case of a Chaplygin gas), a planar wave between two states is tangent to the characteristic cones associated with both states, because it is characteristic.
- If it is genuinely nonlinear, we know from the Lax shock condition that the plane supporting the wave intersects the characteristic cone associated with the downstream state transversally. On the contrary it does not intersect the characteristic cone associated with the upstream state.

We therefore may expand the constant states U^j, $\hat{U}^k$ and the planar wave W^{jk} until the latter meets the cone $\mathcal{C}(\hat{U}^k)$. A comparison with the regular shock reflexion in gas dynamics suggests that the wave bends beyond this intersection, in such a way that it matches further to W^{ji} (remember that in the linear case, W^{jk} and W^{ji} coincide). On its external side, the wave is bounded by the constant state U^j. In general, the bent wave is a free boundary, where the conservation laws are completed by the Rankine–Hugoniot relations. But if the system is linearly degenerate, the bent wave is characteristic with respect to U^j, thus coincides with $\mathcal{C}(U^j)$.

We emphasize that the solution should equal $\hat{U}^k$ between W^{ik}, W^{jk} and $\mathcal{C}(\hat{U}^k)$. In particular, the domain where $U \equiv \hat{U}^k$ is known explicitly. This resembles a lot the situation of the uniform region behind a regular shock reflexion in gas dynamics. This is also consistent with the discussion in Paragraph 4.3 of [10].

Plan of the paper. The rest of the paper is dedicated to the situation for irrotational gas dynamics (because we wish that $m = 2$). Facts about this classical model are gathered in Paragraph 2.1. We describe in Sec. 2 the mathematical problem associated with the construction of the triple interaction. In particular, we show the necessity of a conical wave on a specific example (Theorem 2.1). The conical wave is fully described by a potential obeying a degenerate free-boundary problem over some spherical cap. For a Chapygin equation of state, where this cap is known *a priori*, we prove in Sec. 3 that the boundary-value problem admits a unique solution (Theorem 3.1). The existence and uniqueness of the triple interaction (Theorem 1.1) follows immediately. We recall in Appendix A.1 a few facts about the Chaplygin gas; we show in particular that steady irrotational flows are true flows
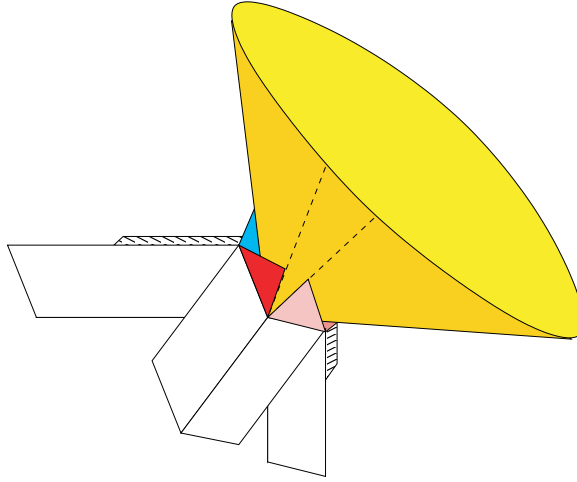
Fig. 4. The conical wave is supported in the interior of the cone displayed on the figure. Planar shocks are taken from Fig. 3. For a Chaplygin gas, this cone is $\mathcal{C}^1$, piecewise quadratic, and tangent to the shocks produced by secondary interactions. In linear acoustics, the conical wave is trivial.

of the gas (Theorem A.1). We recall the calculations underlying the pairwise interaction of planar shocks in Appendix A.2. Finally, we discuss in Appendix A.3 the type of the PDEs governing a steady and self-similar solution of conservation laws.

2. Triple Interaction

We thus focus on gas dynamics, more specifically to isentropic, irrotational flows. The model displays only pressure waves. In a given direction, a pressure wave can be forward and backward, hence $m = 2$. The space dimension is denoted by d.

2.1. *Irrotational flow of a compressible gas*

Let us introduce the velocity potential ψ by $u = \nabla\psi$. The flow is governed by the conservation of mass

$$\partial_t \rho + \mathrm{div}(\rho \nabla \psi) = 0 \tag{2.1}$$

and the Bernoulli equation

$$\partial_t \psi + \frac{1}{2}|\nabla\psi|^2 + \imath(\rho) = 0, \tag{2.2}$$

where

$$\imath(\rho) := \int^\rho \frac{c^2(\mu, \bar{s})}{\mu}\, d\mu.$$

The jump relations for an irrotational flow consist of

- The continuity of the potential ψ. This amounts to saying that the tangential component of the velocity is continuous across a pressure discontinuity.

• The mass conservation

$$[\rho(u \cdot \nu - s)] = 0, \tag{2.3}$$

where ν is the unit normal and s the normal velocity of the discontinuity.

Steady flows. In a steady irrotational flow, $\partial_t \psi$ is a constant, though not necessarily zero. We have therefore the Bernoulli equation

$$\frac{1}{2}|\nabla\psi|^2 + \imath(\rho) = \kappa, \tag{2.4}$$

with κ a constant. Because $\imath'$ is positive, this can be resolved in terms of the density:

$$\rho = h\left(\frac{1}{2}|\nabla\psi|^2 - \kappa\right).$$

Then the system of PDEs reduces to a single second-order equation

$$\mathrm{div}\left(h\left(\frac{1}{2}|\nabla\psi|^2 - \kappa\right)\nabla\psi\right) = 0. \tag{2.5}$$

This equation is valid in the distributional sense: according to (2.3), discontinuities of $\nabla\psi$ obey to the jump relation

$$[h(\cdots)\nabla\psi \cdot \nu] = 0.$$

Equation (2.5) is elliptic (respectively hyperbolic) if the flow is subsonic (respectively supersonic), meaning that $|u| < c$ (respectively $|u| > c$).

Steady-self-similar flows. Equation (2.5) is compatible with the constraint that ψ be positively homogeneous of degree one. This is the situation encountered in constant states and also in the conical wave. The corresponding system is

$$\mathrm{div}\left(h\left(\frac{1}{2}|\nabla\psi|^2 - \kappa\right)\nabla\psi\right) = 0, \quad (x \cdot \nabla)\psi = \psi. \tag{2.6}$$

The type is now determined by the restriction of the symbol of (2.5)

$$P(x;\xi) = h(\cdots)|\xi|^2 + h'(\cdots)(\xi \cdot \nabla\psi)^2$$

to the subspace $x^\perp$. The trace of this restriction is $dh + |\nabla\psi|^2 h'$ (that of P over $\mathbb{R}^d$), minus the value of P on the unit normal; it is thus equal to

$$(d-1)h + \frac{|x \times \nabla\psi|^2}{|x|^2} h'.$$

The number h is an eigenvalue of multiplicity $d - 2$, because $P \equiv h|\xi|^2$ over $x^\perp \cap \nabla\psi^\perp$. The remaining eigenvalue of the restriction is thus

$$\lambda := h + \frac{|x \times \nabla\psi|^2}{|x|^2} h'.$$

Because h is positive, the type of the system (2.6) is determined by the sign of λ. Using the obvious fact that $h' = c^2 h$, we therefore conclude that the system for conical waves is hyperbolic at x if $c^2|x|^2 < |x \times \nabla \psi|^2$ and elliptic if $c^2|x|^2 > |x \times \nabla \psi|^2$. In particular, the characteristic cone $\mathcal{C}(U)$ has equation

$$c(\rho)^2 |x|^2 = |x \times u|^2. \tag{2.7}$$

This is significantly different from the usual self-similar situation, where the characteristic cone has an equation $|u - \frac{x}{t}|^2 = c(\rho)^2$.

2.2. *Obstruction to a piecewise constant solution*

We assume from now on that d equals 3 and the gas obeys the Chaplygin equation of state with $a = 1$:

$$p = p_0 - \frac{1}{\rho}, \quad c = \frac{1}{\rho}.$$

Let us recall that our data is made of three steady shocks passing through the origin and separating a central state U_0 from states U_1, U_2, U_3. The plane between U_0 and U_j is denoted Π_j, its unit normal oriented toward U_j being ν_j. Each of these shocks is backward, meaning that $U_j \cdot \nu_j$ and $U_0 \cdot \nu_j$ are positive for $j = 1, 2, 3$.

We treat first the symmetric case in a Chaplygin gas, where the data is covariant under a rotation of angle $2\pi/3$. It is enough to start with shocks for which each Π_j is supported by the coordinate plane $\{x_i = x_k = 0\}$ (as usual, i, j, k are distinct indices). Hence ν_j is the jth element of the canonical basis. Say that $u_0 = (1, 1, 1)$, so that $u_0 \cdot \nu_j = 1$. Because the shocks are characteristic, $\rho_0 = 1$. By symmetry, we have $\rho_1 = \rho_2 = \rho_3$, which we simply denote ρ. Then

$$u_1 = \begin{pmatrix} v \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ v \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ v \end{pmatrix},$$

with $v = 1/\rho$ by the shock relation $u \cdot \nu = c$.

Theorem 2.1. *The symmetric triple interaction of* 3D *planar shock waves for a Chaplygin gas does not have a piecewise constant solution made only of primary and secondary planar interactions.*

In other words, the solution to this problem differs qualitatively from that of the same problem in linear acoustics.

To prove this result, we just calculate explicitly the primary and secondary interactions.

Primary interactions. The shocks U_0/U_1 and U_0/U_2 interact along the x_3-axis. By symmetry, the outgoing shocks have normals

$$\nu_1^3 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \nu_2^3 = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

for some angle $\theta \in (\pi/4, 3\pi/4)$. The transmitted state U^3 has velocity $(w, w, 1)^{\mathrm{T}}$. Actually, the symmetry implies that the same angle θ occurs for each of the primary interactions, and we have

$$u^1 = \begin{pmatrix} 1 \\ w \\ w \end{pmatrix}, \quad u^2 = \begin{pmatrix} w \\ 1 \\ w \end{pmatrix}, \quad u^3 = \begin{pmatrix} w \\ w \\ 1 \end{pmatrix},$$

for some w, to be determined. Finally, we have $\rho^1 = \rho^2 = \rho^3$, which we denote r. An elementary use of $u \cdot \nu = 1/\rho$ gives that θ is the nonzero[c] solution of

$$v \cos \theta + \sin \theta = \frac{1}{\rho} \quad \text{or} \quad \cos \theta + \rho \sin \theta = 1.$$

This yields

$$\theta = 2 \arctan \rho. \tag{2.8}$$

Meanwhile

$$w = \frac{\cos \theta - v \sin \theta}{\cos \theta - \sin \theta} = \frac{1}{\sin \theta - \cos \theta}, \quad r = \frac{1}{w(\cos \theta + \sin \theta)} = \frac{\sin \theta - \cos \theta}{\cos \theta + \sin \theta}.$$

We point out that, thanks to (2.8), we may parametrize the symmetric data $U_{1,2,3}$ by either ρ or θ.

Secondary interactions. The shocks U_3/U^1 and U_3/U^2 interact along the line perpendicular to both ν_3^1 and ν_3^2. Its direction is L_3, as well as the lines associated with the other secondary interactions, are thus

$$L_1 = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ \cos \theta \end{pmatrix}, \quad L_2 = \begin{pmatrix} \cos \theta \\ -\sin \theta \\ \cos \theta \end{pmatrix}, \quad L_3 = \begin{pmatrix} \cos \theta \\ \cos \theta \\ -\sin \theta \end{pmatrix}.$$

If the solution of the triple interaction was piecewise constant, made only of primary and secondary interactions, then the remaining shocks would be supported by the planes passing through two of the lines above. For instance, one of these waves would be supported by the plane P^3 spanned by L_1 and L_2, which has equation

$$(x + y) \cos \theta = z(\cos \theta - \sin \theta).$$

This plane, supporting a shock between U^3 and the eighth state, would have to be characteristic for U^3. This means

$$2w \cos \theta - \cos \theta + \sin \theta = \frac{1}{r} \sqrt{1 + 2(\cos^2 \theta - \cos \theta \sin \theta)},$$

or equivalently

$$2 \cos \theta + (\cos \theta - \sin \theta)^2 = (\cos \theta + \sin \theta)\sqrt{1 + 2(\cos^2 \theta - \cos \theta \sin \theta)}.$$

[c]At first glance $\theta_0 = 0$ is an obvious solution of our jump relations, but it corresponds to the incoming shock.

This is clearly false (take for instance $\theta = \pi/3$), except for finitely many values of the parameter[d] θ. Therefore P^3 is not characteristic for U^3, and the proof of Theorem 2.1 is complete.

<div align="right">Q.E.D.</div>

2.3. *The conical wave resolves the obstruction*

To resolve this obstruction, we follow the guidelines described in Paragraph 1.3.1. The solution is obtained by solving primary and secondary planar interactions and then solving an elliptic FBVP. The solution of the latter is the conical wave.

The general picture. The conical wave is described by a solution of (2.6) in some conical domain Λ. Its boundary and the boundary conditions are of two kinds. In an outer neighborhood of Λ, the flow is piecewise constant and equal to $U_1, U^3, U_2, U^1, U_3, U^2$ in cyclic order. The boundary of Λ is thus the union of six conical surfaces $\Sigma_{1,2,3}$ and $\Sigma^{1,2,3}$. Each of these manifolds may be either characteristic or non-characteristic. An analogy with the situation in the regular reflexion of a shock against a wedge (see [10]) suggests that a Σ_j is a shock, non-characteristic for a genuinely nonlinear equation of state, and that a Σ^j is a sonic locus. If this is true (let us say that this is a conjecture), then $\Sigma^j = \mathcal{C}(U^j)$ is explicitly known; according to (2.7), its equation is

$$|u^j \times x|^2 = c(\rho^j)^2 |x|^2. \tag{2.9}$$

On the contrary, the *reflected* shock Σ_j is a free boundary. The planar shock U_j/U^k ends where it meets Σ^j; then the component Σ_j begins. Thus the lines $L_{1,2,3}$ described in the previous section[e] are the edges of a conical domain whose section is not a triangle (forbidden by Theorem 2.1), but is made of six segments (corresponding to the shocks U_j/U^k) and three reflected shocks. This situation is described in Fig. 5.

The PDE inside the conical wave. We use spherical coordinates (r, ω), with $r = |x|$ and $\omega = x/r$. We write that the state U depends only upon ω. At the level of the potential, this means $\psi(x) = r\theta(\omega)$ for some function θ. We denote D the spherical section of the domain: $\Lambda =]0, +\infty[\times D$.

The variational formulation of (2.5) is

$$\int_\Lambda h\left(\frac{1}{2}|\nabla\psi|^2 - \kappa\right)\nabla\psi \cdot \nabla\chi \, dx = 0, \quad \forall \chi \in \mathcal{D}(\Lambda). \tag{2.10}$$

Because $\mathcal{D}(\Lambda) = \mathcal{D}(]0 + \infty[) \otimes \mathcal{D}(D)$, it is enough to choose test functions of the form $f(r)g(\omega)$. With $|\nabla\psi|^2 = \theta^2 + |\nabla_\omega\theta|^2$ and

$$\nabla\psi \cdot \nabla\chi = \theta f' g + \frac{f}{r}\nabla_\omega\theta \cdot \nabla_\omega g,$$

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[d] Among which $\theta = \pi/2$ corresponds to the trivial case of a constant data/solution.
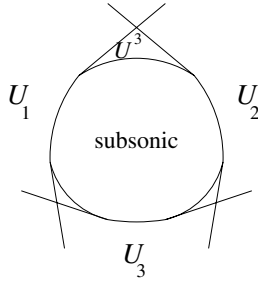[e] These lines are important for a general gas too.

Fig. 5.   The influence domain of the origin in a section $x_1+x_2+x_3 = 1$. The flow is uniform, equal to $U^{1,2,3}$, in three small triangles bordered by sonic circles. It is not uniform in the inner convex domain. On the exterior of the big triangle, the flow is uniform, equal to $U_{1,2,3}$. In general, the sonic circles are not tangent to the boundary of the influence domain, which is only piecewise $\mathcal{C}^1$.

we find

$$\int_0^{+\infty} r^2 dr \int_D h\left(\frac{\theta^2 + |\nabla_\omega \theta|^2}{2} - \kappa\right)\left(f'\theta g + \frac{f}{r}\nabla_\omega \theta \cdot \nabla_\omega g\right)d\omega = 0$$

for all test f and g. Integrating by parts in r, we see that this formulation does not depend on f at all, and reduces to

$$\int_D h\left(\frac{\theta^2 + |\nabla_\omega \theta|^2}{2} - \kappa\right)(-2\theta g + \nabla_\omega \theta \cdot \nabla_\omega g)d\omega = 0, \quad \forall\, g \in \mathcal{D}(D).$$

Equation (2.6) is therefore equivalent to the following second-order PDE over the spherical domain D:

$$\operatorname{div}_\omega(h(\cdots)\nabla_\omega \theta) + 2\theta h\left(\frac{\theta^2 + |\nabla_\omega \theta|^2}{2} - \kappa\right) = 0, \quad \omega \in D, \qquad (2.11)$$

where the divergence operator stands for the adjoint of $-\nabla_\omega$ over the unit sphere.

Because ψ has to be continuous, one of the boundary conditions for (2.11) expresses the continuity of θ across ∂D:

$$\theta|_{\partial D} = \theta|_{\text{out}}. \qquad (2.12)$$

In addition, and because θ has to satisfy (2.11) over the whole sphere in the sense of distribution, we also have

$$h\frac{\partial \theta}{\partial \nu}\bigg|_{\partial D} = h\frac{\partial \theta}{\partial \nu}\bigg|_{\text{out}}. \qquad (2.13)$$

Because the outer flow is piecewise constant, with $U \equiv \bar{U}$ in any component away from the planes Π^{jk}, the right-hand sides in (2.12) and (2.13) are locally given by

$$\theta|_{\text{out}} = \omega \cdot \bar{u}, \quad h\frac{\partial \theta}{\partial \nu}\bigg|_{\text{out}} = \bar{\rho}\bar{u} \cdot \nu.$$

As mentioned above, some parts of the boundary are known and characteristic. There (2.12) and (2.13) are redundant, and it is enough to write (2.12) only. On the

contrary, other parts of ∂D are free boundaries and both (2.12), (2.13) are needed to make a well-posed FBVP.

The picture for a Chaplygin gas. Under a Chaplygin equation of state, all the boundaries Σ_j and Σ^j are characteristic (this is not any more a conjecture). For instance, the equation of Σ_j is

$$|u_j \times x|^2 = c(\rho_j)^2 |x|^2. \tag{2.14}$$

The boundary is now explicit, and it is sufficient to pose the boundary condition (2.12). For the sake of completeness, the following lemma tells that all the relevant waves match nicely (see Fig. 6).

Lemma 2.1. *Let us assume a Chaplygin equation of state. Then the cones Σ_j and Σ^k intersect the planar shock W_j^k along the same line L^{jk}. Along this line, these three varieties are tangent to each other.*

Proof. Let ν be normal to the plane, whose equation is thus $x \cdot \nu = 0$. We recall that $u_j \cdot \nu = c_j$, $u^k \cdot \nu = c^k$ and $u_j \times \nu = u^k \times \nu$. Let us denote u_T the common tangential component:

$$u_T = u_j - c_j \nu = u^k - c^k \nu.$$

On the intersection of the plane and of one of the cones (we write u, c for the state), we have $x \cdot \nu = 0$ and $|x \times u|^2 = c^2 |x|^2$. Because x is a tangent vector, $x \times u_T$ is normal whereas $x \times \nu$ is tangent. This implies $|x \times u|^2 = |x \times u_T|^2 + c^2 |x \times \nu|^2$, and therefore $|x \times u_T| = 0$. Finally, the cone intersect the plane along the line spanned by u_T. Because the intersection is one line only, it is a tangential intersection. $\qquad\square$

The magnitude of the conical wave. The magnitude of the conical wave must be rather small when the amplitudes of the steady shocks in the data are small. On the one hand, is the question of how much the section of the influence domain of the origin deviates from an exact triangle. On the other hand, is what is the amplitude of the conical wave, in terms of this deviation (recall that linear potentials are exact solutions of the PDEs). We leave these questions for a future work.
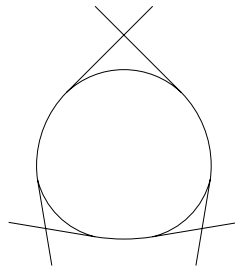

Fig. 6. The influence domain for a Chaplygin gas. The sonic circles are tangent to its boundary, which is now of class $\mathcal{C}^1$, piecewise $\mathcal{C}^2$.

3. Existence of the Conical Wave for a Chaplygin Gas

We restrict now to the equation of state of Chaplygin/von Kármán. The construction of the conical wave reduces to that of the potential θ, where $\psi(x) = r\theta(\omega)$ and $u = \nabla\psi$, whereas

$$\rho = h\left(\frac{1}{2}|\nabla\psi|^2 - \kappa\right) = \frac{1}{\sqrt{|\nabla\psi|^2 - 2\kappa}}.$$

The domain Λ is bounded by six conical pieces of circular cones. Because of Lemma 2.1, Λ is convex and its boundary is $\mathcal{C}^1$ away from the origin.

3.1. *The boundary-value problem*

From (2.11), we need to solve

$$\operatorname{div}_\omega \frac{\nabla_\omega\theta}{\sqrt{\theta^2 + |\nabla_\omega\theta|^2 - 2\kappa}} + \frac{2\theta}{\sqrt{\theta^2 + |\nabla_\omega\theta|^2 - 2\kappa}} = 0, \quad \omega \in D, \tag{3.1}$$

where D is the spherical section of Λ. The boundary condition is (2.12). Because θ_{out} is locally of the form $u \cdot \omega$ where u is a constant state, and ∂D has locally an equation $c^2 = (\omega \times u)^2$, (2.12) becomes $\theta = \sqrt{|u_{\text{out}}|^2 - c_{\text{out}}^2}$ (we shall see below that θ is positive over ∂D). Equivalently, we have

$$\theta = \sqrt{2\kappa} \quad \text{over } \partial D. \tag{3.2}$$

We recall that (2.13) is redundant with (3.2): because of $\sqrt{\theta^2 + |\nabla_\omega\theta|^2 - 2\kappa} = |\nabla_\omega\theta|$, (2.13) only says that the vectors $\nabla_\omega\theta$ on both sides of ∂D are colinear. But this is an obvious consequence of the fact that ∂D is a level line of θ.

In terms of the potential, (2.6.1) can be written

$$(|\nabla\psi|^2 - 2\kappa)\Delta\psi - \mathrm{D}^2\psi(\nabla\psi, \nabla\psi) = 0, \quad x \in \Lambda, \tag{3.3}$$

whereas (2.6.2) writes

$$\mathrm{D}^2\psi\, x = 0. \tag{3.4}$$

Equation (3.1) is hyperbolic or elliptic whenever θ^2 is less or larger than 2κ. Because we are looking for a solution that is not piecewise constant, we rule out the hyperbolic regime and impose instead

$$\theta > \sqrt{2\kappa}. \tag{3.5}$$

3.2. *The maximum principle*

Although Eq. (2.6.1) is quasilinear of the form

$$\sum_{\alpha,\beta} a_{\alpha\beta}(\nabla\psi)\partial_\alpha\partial_\beta\psi = 0,$$

it is unclear whether it satisfies a maximum principle whenever it is elliptic. The reason is that we are looking for solutions that are homogeneous of degree one over a conical domain; therefore the difference $\psi_+ - \psi_-$ between a sub- and a super-solution is linear along every ray and cannot achieve a maximum or a minimum, unless it vanishes identically. Using the homogeneous parts $\theta_\pm := r^{-1}\psi_\pm$ does not help: at a minimum or a maximum of $\theta_+ - \theta_-$ (or of θ_+/θ_-), $\nabla\psi_+$ and $\nabla\psi_-$ do not coincide in general, and we do not obtain the equality of coefficients $a_{\alpha\beta}$.

Despite these difficulties, it turns out that the maximum principle holds true, although in a rather subtle way. Up to a rescaling, we may assume that $2\kappa = 1$. Then our problem is elliptic whenever $\theta > 1$, that is $\psi > r$. The appropriate quantity to look at is the function $z > 0$ defined implicitly by

$$\psi(x) =: r \cosh z(x), \quad z > 0.$$

We therefore deal with functions z that are homogeneous of degree zero. In other words, we have *a priori* $\partial_r z \equiv 0$. It is straightforward to verify that ψ is a solution of (3.3) if and only if z is a homogeneous solution of

$$(r^2|\nabla z|^2 + 1)\Delta z - r^2 \mathrm{D}^2 z(\nabla z, \nabla z) + \frac{2}{\tanh z}\left(|\nabla z|^2 + \frac{1}{r^2}\right) = 0. \tag{3.6}$$

The boundary condition (3.2) becomes

$$z = 0 \quad \text{over } \Gamma, \tag{3.7}$$

which makes the lower order term in (3.6) singular.

Let us write (3.6) in the quasilinear form

$$\sum_{\alpha,\beta} a_{\alpha\beta}(x, \nabla z)\partial_\alpha\partial_\beta z + N(x, z, \nabla z) = 0, \tag{3.8}$$

where the principal part is elliptic. The lower-order correction $z \mapsto N(x, z, p)$ is non-increasing in z, and the coefficients $a_{\alpha\beta}$ depend upon ∇z but not z itself. Such an equation satisfies the maximum principle, in the sense that if z_- is a sub-solution and z_+ is a super-solution of the PDE over a domain Ω, with $z_- \leq z_+$ over $\partial\Omega$, then there happens $z_- \leq z_+$ everywhere in Ω. Because $\theta = \cosh z$, and the hyperbolic cosine is increasing over $(0, +\infty)$, this property translates in terms of θ. We therefore obtain the following statement.

Proposition 3.1. *Let D_1 be a bounded open domain in the unit sphere. Let $\theta_\pm \in W^{1,\infty}(\overline{D_1})$ be two functions satisfying $\theta_\pm > \sqrt{2\kappa}$ in D_1. Let us assume that $\theta_\pm$ are sub- and super-solutions of (3.1):*

$$\mathrm{div}_\omega \frac{\nabla_\omega \theta_+}{\sqrt{\theta_+^2 + |\nabla_\omega \theta_+|^2 - 2\kappa}} + \frac{2\theta_+}{\sqrt{\theta_+^2 + |\nabla_\omega \theta_+|^2 - 2\kappa}}$$

$$\leq 0 \leq \mathrm{div}_\omega \frac{\nabla_\omega \theta_-}{\sqrt{\theta_-^2 + |\nabla_\omega \theta_-|^2 - 2\kappa}} + \frac{2\theta_-}{\sqrt{\theta_-^2 + |\nabla_\omega \theta_-|^2 - 2\kappa}}$$

in the sense of distributions. Finally, let us assume that $\theta_- \leq \theta_+$ on ∂D_1.
Then $\theta_- \leq \theta_+$ in D_1.

3.2.1. *Other nonlinearities*

When a positive function $q \mapsto a(q)$ is given, we expect that the PDE

$$\operatorname{div}(a(|\nabla\psi|)\nabla\psi) = 0 \tag{3.9}$$

admits solutions that are homogeneous of degree one (conical waves). Because linear maps $x \mapsto \ell \cdot x$ fit this homogeneity and are exact solutions, they may serve as sub- or super-solutions for the corresponding BVP. They will therefore provide lower/upper bounds, whenever the maximum principle is valid.

It seems however that the strategy adopted hereabove works only for a very few functions a. So far, we applied it successfully only when the nonlinearity is of the form

$$a_\alpha(q) = (q^2 - q_0^2)^{-\alpha/2},$$

where $q_0 > 0$ and α are constants. When $\psi > q_0$ (meaning ellipticity of the homogeneous problem), we introduce the change of variable $\psi = q_0 r \cosh(\gamma z)$, with γ a positive constant. The corresponding PDE is

$$(|\nabla\psi|^2 - q_0^2)\Delta\psi - \alpha \mathrm{D}^2\psi(\nabla\psi, \nabla\psi) = 0.$$

We point out that the choice $\alpha = 0$ corresponds to the linear equation $\Delta\psi = 0$.

If ψ is homogeneous of degree one, then it is a solution of (3.9) if and only if z is homogeneous of degree zero and is a solution of

$$\Delta z - \frac{\alpha\gamma^2 r^2}{\gamma^2 r^2 |\nabla z|^2 + 1} \mathrm{D}^2 z(\nabla z, \nabla z) + \left((1-\alpha)\gamma^2 |\nabla z|^2 + \frac{2}{r^2} \right) \frac{1}{\tanh z} = 0. \tag{3.10}$$

This is again an equation of the form (3.8), with N non-increasing in z and an elliptic principal part if $\alpha \leq 1$. Therefore it satisfies a maximum principle. For instance, if $z_\pm$ are sub- and super-solutions of (3.10) in the cone Λ, then $z_+ - z_-$ cannot achieve a non-negative minimum in the interior of Λ, unless $z_+ \equiv z_-$. Thus if $z_\pm$ are homogeneous of degree zero, and if z_- is less than or equal to z_+ on the boundary of Λ, then $z_- \leq z_+$ everywhere in Λ. The latter property translates in terms of θ. If $\theta_\pm$ are sub- and super-solutions of

$$\operatorname{div}_\omega(a_\alpha(\sqrt{\theta^2 + |\nabla_\omega\theta|^2})\nabla_\omega\theta) + 2\theta a_\alpha(\sqrt{\theta^2 + |\nabla_\omega\theta|^2}) = 0, \tag{3.11}$$

that are greater than q_0 in Λ, and if $\theta_- \leq \theta_+$ over ∂D, then this remains true in D. In other words, Proposition 3.1 applies to the equation with nonlinearity a_α.

Remarks

- The nonlinearity a_α corresponds to an equation of state of the form (at constant entropy)

$$p_\alpha(\rho) = p_0 - a^2 \rho^{1-2/\alpha},$$

with p_0 and a constants. Equivalently,

$$\rho = \left(\frac{a}{p_0 - p}\right)^{\frac{\alpha}{2-\alpha}}.$$

If $\alpha = 0$, this means that ρ is constant; this is the incompressible case.

- In the linear case ($\alpha = 0$), the maximum principle is equivalent to the existence of a positive subsolution of the homogeneous Dirichlet boundary-value problem. Therefore the conclusion followed directly from the assumption that θ_+ is positive. When the first eigenvalue of the Laplacian with Dirichlet boundary condition is not greater than 2, our statement is void, because there does not exist such a θ_+.

3.2.2. *Relation with minimal surfaces*

This paragraph is an observation made by Lihe Wang (Univ. of Iowa). Equation (3.6) can be written

$$\operatorname{div}_\omega \frac{\nabla_\omega z}{\sqrt{1 + |\nabla_\omega z|^2}} + \frac{2\cosh z}{\sinh z \sqrt{1 + |\nabla_\omega z|^2}} = 0.$$

This is the Euler–Lagrange equation of the functional

$$\mathcal{L}[z] := \int_D \frac{\sqrt{1 + |\nabla_\omega z|^2}}{\sinh^2 z}\, d\omega.$$

The latter is the area of the graph $x_3 = z(\omega)$ when $\mathbf{S}^2 \times (0, \infty)$ is endowed with the metric

$$ds^2 = \frac{d\sigma^2 + dx_3^2}{\sinh^2 x_3},$$

where $d\sigma$ is the standard metric on the sphere. This Riemannian manifold is complete, with negative scalar curvature[f] $-5 - \cosh^2 x_3$. Its infinity is $(\mathbf{S}^2 \times \{0\}) \cap \{\infty\}$. Our boundary condition being $z|_{\partial D} = 0$, it is equivalent to a prescribed asymptotics at infinity: we are therefore looking for a *complete* minimal surface.

3.3. *First a priori estimates*

We begin with L^∞-estimates, using sub- and super-solutions. We exploit the fact that linear functions $x \mapsto w \cdot x$ solve the PDE (3.3). The corresponding $\theta^w : \omega \mapsto$

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[f]Lihe Wang, personal communication.

$w \cdot \omega$ is thus a solution of (3.1). Let us consider the sets

$$W_+ := \{w \in \mathbb{R}^3 \,|\, \theta^w > \sqrt{2\kappa} \text{ over } \partial D\},$$

$$W_- := \{w \in \mathbb{R}^3 \,|\, \theta^w < \sqrt{2\kappa} \text{ over } \partial D\}.$$

When $w \in W_+$, the function θ^w is larger than $\sqrt{2\kappa}$ over D, because the cone Λ is convex. It is thus a super-solution of the boundary-value problem. The function

$$\theta^+ := \inf\{\theta^w \,|\, w \in W_+\}$$

is again a super-solution: it satisfies

$$\text{div}_\omega \frac{\nabla_\omega \theta^+}{\sqrt{\theta^{+2} + |\nabla_\omega \theta^+|^2 - 2\kappa}} + \frac{2\theta^+}{\sqrt{\theta^{+2} + |\nabla_\omega \theta^+|^2 - 2\kappa}} \leq 0$$

in D and $\theta^+ \geq \sqrt{2\kappa}$ over ∂D. We actually have even more, because Λ is a convex cone: for every $\omega \in \partial D$, there exists a w in W_+ such that $w \cdot x > \sqrt{2\kappa} \,|x|$ over Λ, with equality along the line $\mathbb{R}^+\omega$. We thus have $\theta^+ = \sqrt{2\kappa}$ over ∂D.

According to Proposition 3.1, θ^+ is an upper bound of the expected solution:

$$\theta \leq \theta^+. \tag{3.12}$$

In particular, we obtain a uniform bound over D.

The situation is not much different for sub-solutions. If $w \in W_-$, then θ^w is a sub-solution of the BVP in the subdomain D_w where θ^w is larger than $\sqrt{2\kappa}$. By the maximum principle stated above, we have the estimate $\theta^- \leq \theta$ in D_w, and even in D because otherwise we have $\theta^w \leq \sqrt{2\kappa} \leq \theta$. Finally, we find the lower bound

$$\theta^- := \sup\{\theta^w \,|\, w \in W_-\} \leq \theta. \tag{3.13}$$

Obviously, θ^- is less than or equal to $\sqrt{2\kappa}$ over ∂D. This turns out to be an equality because if $\omega \in D$ is given, we can choose $w := (\sqrt{2\kappa} + \epsilon)\omega$ which belongs to W_- for $0 < \epsilon \ll 1$. This implies $\theta^- > \sqrt{2\kappa}$ in D, hence $\theta^- \geq \sqrt{2\kappa}$ on the boundary. Finally, we have

$$\theta^\pm \equiv \sqrt{2\kappa} \quad \text{over } \partial D. \tag{3.14}$$

Notice that (3.14) is valid for θ^- even if Λ is not convex. The convexity is needed only so far as θ^+ is concerned.

Finally, let us set $q_0 = \sqrt{2\kappa}$ in the nonlinearity a_α. Because Proposition 3.1 applies, and because the functions θ^w are solutions of the PDE for every nonlinearity, we see that a solution $\theta \geq q_0$ of (3.11), such that $\theta = q_0$ on the boundary, satisfies again the estimates (3.12) and (3.13).

3.4. *Uniform ellipticity and Lipschitz estimate*

The previous paragraph gives us a pointwise estimate $\theta^- \leq \theta \leq \theta^+$ in D, which ensures that θ is bounded in L^∞ (because θ^+ is bounded) and that Eq. (3.1) is elliptic in D (because $\theta^- > \sqrt{2\kappa}$ there).

We shall prove in a further section the existence of a solution by an approximation procedure. Because the equation is quasilinear, we need some compactness of the gradient of θ, which will be obtained by the Ascoli–Arzela Theorem and a suitable control of second-order derivatives. The latter control is available whenever

(1) the equation is *uniformly* elliptic,
(2) we have some control of θ and its first-order derivatives.

Notice that this uniformity and this control are needed only locally, that is on compact subsets of D. We emphasize that uniformly ellipticity is not guaranteed for an equation like (3.1) in general. This is a well-known flaw of equations whose principal part ressembles that of the minimal surface equation (see [5, 9]).

Let us for instance assume *a priori* bounds $m < \theta < M$, where $m > \sqrt{2\kappa}$ and $M < \infty$ are given. Such bounds are true over every compact subset of D, thanks to (3.12), (3.13). The ratio of the eigenvalues of the quadratic symbol of (3.1) can be estimated by

$$\frac{\mu_2}{\mu_1} = \frac{\theta^2 + |\nabla_\omega \theta|^2 - 2\kappa}{\theta^2 - 2\kappa} \leq \frac{M^2 + |\nabla_\omega \theta|^2 - 2\kappa}{m^2 - 2\kappa}.$$

The right-hand side cannot be *a priori* bounded, unless we know a bound of the gradient $\nabla_\omega \theta$. Because $|\nabla_x \psi|^2 = \theta^2 + |\nabla_\omega \theta|^2$, we deduce that the uniform ellipticity requires a Lipschitz estimate of ψ.

Lipschitz estimate at the boundary. Because the expected solution θ satisfies $\theta^- \leq \theta \leq \theta^+$ in D, with equalities at the boundary, θ admits a normal derivative, which satisfies

$$\left| \frac{\partial \theta}{\partial \nu} \right| \leq \left| \frac{\partial \theta^+}{\partial \nu} \right|. \tag{3.15}$$

Equivalently, because $|\nabla \psi|^2 = \theta^2 + |\nabla_\omega \theta|^2$ and $\theta \equiv \sqrt{2\kappa}$ at the boundary, we have

$$\left| \frac{\partial \psi}{\partial \nu} \right| \leq \left| \frac{\partial \psi^+}{\partial \nu} \right|, \tag{3.16}$$

with $\psi^+ := r\theta^+$.

Interior Lipschitz estimate. Let us assume that ψ is smooth enough and homogeneous of degree one. Then $|\nabla \psi|$ is homogeneous of degree zero and therefore must achieve its upper bound over $\overline{\Lambda}$, because the section $\overline{D}$ of the latter is compact.

We start with Eq. (3.3), which has the form $A(\nabla \psi) : \mathrm{D}^2 \psi = 0$. Differentiating and taking the scalar product with $\nabla \psi$, we obtain the identity

$$A : \mathrm{D}^2 |\nabla \psi|^2 = 2 \sum_\ell \nabla \partial_\ell \psi^\mathrm{T} A \nabla \partial_\ell \psi + \sum_\ell \left(\frac{\partial A}{\partial p_\ell} : \mathrm{D}^2 \psi \right) \partial_\ell |\nabla \psi|^2.$$

Remember that A is not positive in general. It is positive only on $x^\perp$. Using the fact that $\nabla\psi$ is homogeneous of degree zero, we also have $D^2\psi : x \otimes x \equiv 0$. Combining both identities, we obtain

$$(A + \lambda x \otimes x) : D^2|\nabla\psi|^2 = 2\sum_\ell \nabla\partial_\ell\psi^T A\nabla\partial_\ell\psi + \sum_\ell \left(\frac{\partial A}{\partial p_\ell} : D^2\psi\right)\partial_\ell|\nabla\psi|^2.$$

Because of $(D^2\psi)x = 0$ and the positivity of A over $x^\perp$, we have $\nabla\partial_\ell\psi^T A\nabla\partial_\ell\psi \geq 0$ for every ℓ. This implies a differential inequality

$$(A + \lambda x \otimes x) : D^2|\nabla\psi|^2 + \vec{b}\cdot\nabla|\nabla\psi|^2 \geq 0.$$

We may choose λ large enough that the matrix $A + \lambda x \otimes x$ is positive definite. The inequality is now elliptic and the maximum principle applies: $|\nabla\psi|^2$ cannot achieve a maximum at an interior point, unless it is constant. Therefore its maximum is reached at the boundary, and we obtain the global Lipschitz estimate

$$\|\nabla\psi\|_{L^\infty} \leq \sup_{\omega\in\partial D}\sqrt{2\kappa + \left(\frac{\partial\theta^+}{\partial\nu}(\omega)\right)^2}.$$

Because $|\nabla_\omega\theta|^2 = |\nabla\psi|^2 - 2\kappa$ in D, we conclude that

$$\|\nabla_\omega\theta\|_{L^\infty} \leq \sup_{\omega\in\partial D}\left|\frac{\partial\theta^+}{\partial\nu}(\omega)\right|. \tag{3.17}$$

Regularity up to the boundary. Even with Lipschitz estimates in hand, uniform ellipticity will be achieved only on compact subsets of D, because the equation degenerates at the boundary. Therefore, our regularity estimates (for instance those of second-order derivatives) will be valid on every such compact subset only, contrary to (3.17). In particular, this approach does not provide regularity up to the boundary, even though we believe that full regularity holds true. This question is left for a future analysis.

3.5. *The approximation procedure*

Our strategy is a continuation method. We intend to treat a parametrized BVP, where the nonlinearity a is replaced by one depending smoothly upon a constant $\alpha \in [0,1]$. Because we include the maximum principle, the nonlinearities must be those encountered in Sec. 3.2.1:

$$a_\alpha(q) := (q^2 - 2\kappa)^{-\alpha/2}.$$

The BVP we are interested in corresponds to the choice $\alpha = 1$, while $\alpha = 0$ yields the linear problem of harmonic functions. For a given α, the PDE writes

$$(|\nabla\psi|^2 - 2\kappa)\Delta\psi - \alpha D^2\psi(\nabla\psi, \nabla\psi) = 0. \tag{3.18}$$

We complete the PDE with the boundary condition $\psi = \sqrt{2\kappa}\, r$ over $\partial\Lambda$.

Denoting the solution $\phi_\alpha(r\omega) = r\theta_\alpha(\omega)$, the PDE to be satisfied by θ is

$$\mathrm{div}_\omega(a_\alpha(\sqrt{\theta_\alpha^2 + |\nabla_\omega\theta_\alpha|^2})\nabla_\omega\theta_\alpha) + 2\theta a_\alpha(\sqrt{\theta_\alpha^2 + |\nabla_\omega\theta_\alpha|^2}) = 0, \quad \omega \in D. \quad (3.19)$$

Because $\alpha \leq 1$, it is elliptic whenever $\theta^2 + (1-\alpha)|\nabla\theta|^2 > 2\kappa$. In particular, the property $\theta > \sqrt{2\kappa}$ implies ellipticity. For $\alpha < 1$, the latter condition even implies uniform ellipticity, because the ratio of the eigenvalues of the symbol is bounded *a priori*:

$$\frac{\theta^2 + |\nabla\theta|^2 - 2\kappa}{\theta^2 + (1-\alpha)|\nabla\theta|^2 - 2\kappa} \leq \frac{1}{1-\alpha}.$$

Thus all the approximated BVPs are uniformly ellipticity in the range under consideration. Note finally that the ellipticity is uniform for $\alpha = 1$ too, provided that θ stays in an interval $[\epsilon + \sqrt{2\kappa}, +\infty)$, for some $\epsilon > 0$.

Equation (3.19) is a kind of interpolation between the linear one

$$\Delta\theta_1 + 2\theta_1 = 0, \quad \omega \in D \quad (3.20)$$

and the one we are interested in.

We now consider the boundary-value problem with the data

$$\theta_\alpha|_{\partial D} \equiv \sqrt{2\kappa}. \quad (3.21)$$

Naturally, we look for a solution θ_α that satisfies $\theta_\alpha \geq \sqrt{2\kappa}$ in D.

Pointwise and Lipschitz estimates. Because linear functions $x \mapsto w \cdot x$ are exact solutions of (3.18), we may use the same sub- and super-solutions θ^w as in the previous paragraph. Therefore a solution θ_α, if it exists, does satisfy the inequalities

$$\theta^- \leq \theta_\alpha \leq \theta^+, \quad (3.22)$$

which do not depend on s. Then the same arguments as before give the Lipschitz estimate at the boundary:

$$\left|\frac{\partial\theta_\alpha}{\partial\nu}\right| \leq \left|\frac{\partial\theta^+}{\partial\nu}\right|.$$

Finally, the interior Lipschitz estimate follows

$$\|\nabla_\omega\theta_\alpha\|_{L^\infty} \leq \sup_{\omega \in \partial D}\left|\frac{\partial\theta^+}{\partial\nu}(\omega)\right|. \quad (3.23)$$

The linear BVP ($\alpha = 0$). We now turn to the endpoint BVP (3.20), (3.21), which is elliptic and linear. According to the Fredholm principle, it is uniquely solvable provided 2 is not an eigenvalue of $-\Delta$ over D under the Dirichlet boundary condition.

Let us denote $\lambda_0(D) < \lambda_1(D) \leq \cdots$ with $\lambda_n(D) \xrightarrow{n\to+\infty} +\infty$ the spectrum of this operator. We know that $\lambda_0(D)$ is associated with a positive eigenfunction and that $D \mapsto \lambda_0(D)$ is strictly decreasing, in the sense that if D is a strict subset of D', then $\lambda_0(D') < \lambda_0(D)$.

It turns out that for a half-sphere $\Sigma := \mathbf{S}^2 \cap \{\ell \cdot x > 0\}$, one has $\lambda_0(\Sigma) = 2$, the corresponding eigenfunction being $\omega \mapsto \ell \cdot \omega$. Because our cone Λ is strictly convex, its section D is strictly contained in some half-sphere and therefore $\lambda_0(D)$ is larger than 2. We thus obtain:

Proposition 3.2. *Under the Dirichlet boundary condition, the operator $-\Delta - 2$ over D is invertible. In particular, the BVP (3.20) with the data (3.21) is uniquely solvable.*

We point out that because $\lambda_0(-\Delta - 2) > 0$, the inverse $(-\Delta - 2)^{-1}$ is positive. In other words, if $f \geq 0$, then $(-\Delta - 2)^{-1} f \geq 0$.

Existence for every $\alpha \in [0, 1)$. Let J be the set of parameters $\alpha \in [0, 1)$ such that there exists a solution $\theta_\alpha \geq \sqrt{2\kappa}$ to the BVP (3.19), (3.21). Because each problem satisfies the maximum principle, this solution is unique. Thanks to uniform ellipticity, the regularity theory applies, and we have $\theta_\alpha \in \mathcal{C}^\infty(D) \cap \mathcal{C}^\gamma(\bar{D})$, where γ is the degree of regularity of ∂D. In the specific case of the conical wave in a 3D Riemann problem, $\gamma = 2 - \epsilon$ for every $\epsilon > 0$.

Let (α_m, ϕ_m) be a sequence where $\alpha_m \in J$ and ϕ_m is the solution of the corresponding BVP. We assume that $\alpha_m \to \alpha_\infty \in [0, 1)$. Because the estimates are uniform in α over compact sub-intervals of $[0, 1)$, the sequence ϕ_m is bounded in $\mathcal{C}^\infty(D) \cap \mathcal{C}^\gamma(\bar{D})$, and therefore precompact in $\mathcal{C}^1(\bar{D})$. Therefore, we may pass to the limit in a subsequence, and the limit ϕ_∞ is a $\mathcal{C}^1$ solution of the BVP at parameter α_∞. It satisfies $\theta_\infty \geq \sqrt{2\kappa}$, and thus $\alpha_\infty \in J$. This shows that J is a closed set.

Next, let $\alpha_0 \in J$ be given and ϕ_0 be the solution of the corresponding BVP. We work in terms of the auxiliary unknown z. The equation has the form

$$A(\alpha; x, \nabla z) : \mathrm{D}^2 z + N(\alpha; z, \nabla z) = 0,$$

where $A(\alpha; x, p)$ is positive definite, and $z \mapsto N(\alpha; z, p)$ is non-increasing. The BVP can be rewritten as an abstract problem

$$\mathcal{F}(\alpha, z) = 0.$$

Inverting $\frac{\partial \mathcal{F}}{\partial z}(\alpha_0, z_0)$ amounts to solving the homogeneous Dirchlet problem for the linear equation

$$A(\alpha; x, \nabla z_0) : \mathrm{D}^2 w + b(x) \cdot \nabla w + \frac{\partial N}{\partial z}(\alpha_0; z_0, \nabla z_0) w = \text{R.H.S.},$$

where

$$b_j(x) := \frac{\partial A}{\partial p_j}(\alpha_0; \nabla z_0) : \mathrm{D}^2 z_0 + \frac{\partial N}{\partial p_j}(\alpha_0; \nabla z_0).$$

Because $\partial N / \partial z \geq 0$, the operator on the left-hand side satisfies the maximum principle, and therefore the Dirichlet problem has at most one solution. By the Fredholm principle, it is uniquely solvable.

With this in hands, we may apply the Implicit Function Theorem to our abstract problem. We obtain that for α in a neighborhood of α_0, the solution exists and is a smooth function of α. Therefore J is open.

Being closed, open and not empty, J equals the whole interval $[0, 1)$. This is the sense of the following proposition.

Proposition 3.3. *Let $\alpha \in [0, 1)$ be given. The BVP* (3.19), (3.21) *admits a unique solution $\theta_\alpha \geq \sqrt{2\kappa}$, which satisfies in addition the estimates* (3.22), (3.23).

Passing to the limit as $\alpha \to 1+$. Let D_1 be a relatively compact open subset of D. We recall that θ^- is uniformly bounded away from $\sqrt{2\kappa}$ over D_1 and that $\theta^- \leq \theta_\alpha$. Because ellipticity remains uniform whenever $(\theta, s) \in [\epsilon + \sqrt{2\kappa}, +\infty) \times [0, 1]$, the regularity theory yields estimates of derivatives at every order of θ_α over D_1, that are uniform in α. By compactness, we may extract a sequence $\alpha_n \to 1+$ such that θ_{s_n} converges towards a function θ_1 in $\mathcal{C}^\infty(D)$. Passing to the limit in (3.19), we find that θ_1 solves our PDE (3.1). Passing to the limit, we still have (3.22), which ensures that θ_1 satisfies the required boundary condition. Likewise, we keep (3.23) in the limit. Finally, we may state our main result:

Theorem 3.1. *The boundary-value problem* (3.1), (3.2) *admits a unique solution such that $\theta > \sqrt{2\kappa}$ in D, with regularity*

$$\theta \in \mathcal{C}^\infty(D) \cap Lip(\bar{D}).$$

This completes the proof of Theorem 1.1.

Other equations of state. When the gas obeys another equation of state, for instance that of the perfect gas, we have seen in Sec. 2 that three among the six pieces of ∂D are free-boundaries. We may expect to solve the free-boundary value problem by following the ideas developed in [1]. We warn the reader that we do not have any more a maximum principle. This flaw might be compensated by a smallness assumption on the data. We leave the existence/uniqueness question open.

Appendices

A.1. *Chaplygin gas*

A real gas is described by its mass density and its entropy s. The Chaplygin equation of state is

$$p(\rho, s) = g(s) - \frac{a(s)^2}{\rho}, \quad a > 0. \tag{A.1}$$

We point out that two extreme regimes are excluded *a priori* for physical reasons. On the one hand, a small density yields a negative pressure. On the other hand, the pressure saturates at very high densities, a fact which is responsible for concentration of mass along codimension-one subsets. Therefore the Chaplygin equation

of state is used only so far as the density remains in a suitable compact interval of $(0, +\infty)$.

The sound speeds is $c = \sqrt{\partial p/\partial \rho} = a/\rho$. Because $\partial(\rho c)/\partial \rho \equiv 0$, the pressure fields are linearly degenerate, in the terminology of hyperbolic systems of first-order equations. This means that shocks are reversible and characteristic.

It was shown in [9] that if the initial data of the 2D Euler equations has a constant entropy, and if the solution is piecewise smooth (discontinuities across hypersurfaces are allowed), then the flow remains isentropic, contrary to what happens for a general equation of state. The proof is actually valid in every spatial dimension. We may therefore focus on isentropic flows even in 3D. Then both g and a are constants in (A.1).

In a steady flow, a pressure discontinuity satisfies (see for instance [9])

$$j^2 = a^2, \quad j := \rho u_- = \rho u_+. \tag{A.2}$$

This gives on both sides of the wave

$$(u_\pm \cdot \nu - s)^2 = a^2/\rho_\pm^2, \tag{A.3}$$

which confirms that pressure waves are sonic: $|u \cdot \nu - s| = c$.

Irrotational flows are true flows. For general equations of state, there is no reason why an irrotational data would produce an irrotational flow beyond the formation of discontinuities. However, the situation is much better for a Chaplygin gas if the flow is steady and isentropic:

Theorem A.1. *Consider a steady shock wave in a Chaplygin gas. Let Γ be the surface of discontinuity and ν its unit normal. Let us decompose the vorticity into normal and tangential components along the shock locus*:

$$\omega = \omega_T + (\omega \cdot \nu)\nu.$$

Then we have the jump relations

$$[\omega \cdot \nu] = 0, \quad \left[\frac{\omega_T}{\rho}\right] = 0. \tag{A.4}$$

In particular, if the flow is irrotational on one side of the shock, it is irrotational on the other side.

Proof. The first equality in (A.4) follows from the differential equation $\operatorname{div} \omega = 0$.

Let $j := \rho u \cdot \nu$ be the net mass flux across Γ. It has the same value on both sides of Γ and is nonzero (it should equal to zero on a slip line). We start from the conservation of momentum, which can be rewritten on each side as

$$(u \cdot \nabla)u - \nabla \frac{a^2}{2\rho^2} = 0.$$

This is equivalent to

$$\nabla \frac{1}{2} \left(|u|^2 - \frac{a^2}{\rho^2} \right) = u \times \omega.$$

If τ be a tangent vector field to Γ, we deduce

$$\tau \cdot \nabla \frac{1}{2} \left(|u|^2 - \frac{a^2}{\rho^2} \right) = \det(\tau, u, \omega).$$

This is valid on each side of Γ. Let us make the difference between both sides. We get

$$\tau \cdot \nabla \frac{1}{2} \left[|u|^2 - \frac{a^2}{\rho^2} \right] = [\det(\tau, u, \omega)].$$

From (A.3), we have $(u \cdot \nu)^2 - a^2/\rho^2 = 0$ on either sides of Γ. We also know that the tangential component of the velocity is continuous across Γ. All this implies that $|u|^2 - a^2/\rho^2$ is continuous too. Because $\tau \cdot \nabla$ is a derivative along Γ, we deduce

$$[\det(\tau, u, \omega)] = 0.$$

Let us decompose as well

$$u = u_T + (u \cdot \nu)\nu = u_T + \frac{j}{\rho} \nu.$$

We have

$$[\det(\tau, u_T, (\omega \cdot \nu)\nu)] + j \left[\det \left(\tau, \frac{\nu}{\rho}, \omega_T \right) \right] + [\det(\tau, u_T, \omega_T)] = 0.$$

The first term vanishes because both u_T and $\omega \cdot \nu$ are continuous across Γ. The third one vanishes too because the three vectors are tangent to Γ thus coplanar. Because $j \neq 0$, there remains

$$\left[\det \left(\tau, \nu, \frac{\omega_T}{\rho} \right) \right] = 0, \quad \text{that is} \quad \det \left(\tau, \nu, \left[\frac{\omega_T}{\rho} \right] \right) = 0.$$

Because τ is an arbitrary tangent vector, $[\omega_T/\rho]$ is orthogonal to the arbitrary tangent vector $\tau \times \nu$. Because it is itself a tangent vector, it must vanish. ☐

In conclusion, piecewise smooth irrotational steady flows are not only good approximations of Euler flows, but they are genuine flows of the full gas with a Chaplygin equation of state.

We end this paragraph with the calculation of the function h for a Chaplygin gas. We have $\imath' = p'/\rho = a^2/\rho^3$, and therefore $\imath = -a^2/(2\rho^2)$, up to an irrelevant additive constant. This gives

$$h(z) = \frac{a}{\sqrt{2z}}. \tag{A.5}$$

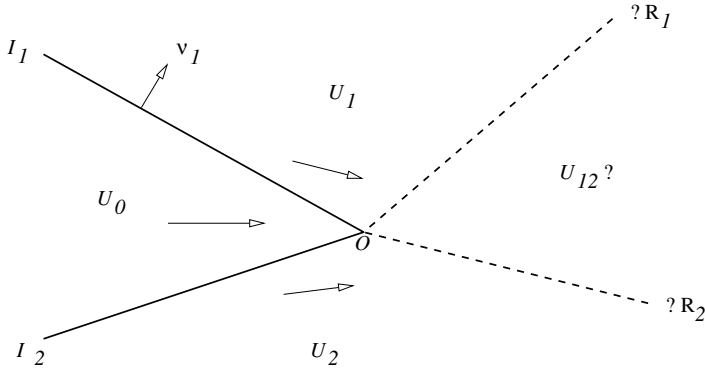From now on, we set $a = 1$, without loss of generality.

Fig. 7.   The binary interaction problem.

A.2. *Pairwise interaction of planar shock waves*

Let us consider an interaction illustrated in Fig. 7. Two incoming planar shocks meet at the origin. Because $m = 2$, we expect that the picture is completed by two emerging shocks, represented by the dashed lines. We are looking for a piecewise constant steady solution of Eq. (2.5). The data is made of the states $U_{0,1,2}$ and the directions $I_{1,2}$, which satisfy the jump conditions (Rankine–Hugoniot and entropy condition). The unknowns are the transmitted state U_{12} and the directions $R_{1,2}$.

We focus on irrotational gas dynamics. Because we are interested in gas dynamics, we assume that $I_{1,2}$ are backward shock waves. This means that the gas flows from the domain labeled 0 into the domains labeled $1, 2$. We point out that if the shock strengths are weak, then the normal velocities $U_0 \cdot \nu_{1,2}$ across shocks are close to the sound speed c_0, and therefore U_0 is highly supersonic:

$$|U_0| \sim c_0 \sqrt{\frac{2}{1 - \cos \alpha}} > c_0,$$

where α is the angle between I_1 and I_2. Equation (2.5) is therefore hyperbolic in the direction of the flow, and this interaction problem is nothing but a one-dimensional Riemann problem. It can be solved by algebraic calculations involving the Rankine–Hugoniot conditions; this is called *shock polar analysis* and is described by Courant and Friedrichs [2] and Dafermos [3], Sec. 17.2-3. The main tool for this calculation is the *Hugoniot locus* associated with a given state U_-. We recall its construction below for the sake of completeness.

Before going further, we emphasize that a pairwise interaction of planar shock waves occurs in every dimension $d \geq 2$. The d-dimensional picture can always be reduced to the case $d = 2$, because among the jump conditions is the fact that the component of the velocity u tangential to the shock is continuous. Therefore a d-dimensional interaction is nothing but the superposition of a 2D interaction and of a constant velocity in the direction of the axis along which the interaction takes

place. Figure 7 can be viewed as the projection of the whole picture, parallel to this axis; the image of the axis is then the point O.

Hugoniot locus for irrotational gas dynamics. It is defined as the set of states U_+ which can be linked to U_- by a steady discontinuity. In order to make it explicit, we write the jump conditions

$$[u \times \nu] = 0 \quad \text{and} \quad [\rho u \cdot \nu] = 0. \tag{A.6}$$

Recall that we have $u_- \cdot \nu > 0$, and therefore we must have $u_+ \cdot \nu > 0$ as well. We denote z_+ the latter, unknown, quantity. From (A.6.2), we have

$$h\left(\frac{1}{2}|u_+|^2 - \kappa\right)z_+ = h\left(\frac{1}{2}|u_-|^2 - \kappa\right)u_- \cdot \nu.$$

With (A.6.1), this is rewritten in the form

$$h\left(\frac{1}{2}(|u_-|^2 - (u_- \cdot \nu)^2 + z_+^2) - \kappa\right)z_+ = h\left(\frac{1}{2}|u_-|^2 - \kappa\right)u_- \cdot \nu. \tag{A.7}$$

Let $F(z_+)$ denote the left-hand side of (A.7). One has

$$F'(z) = h(\cdots) + z^2 h'(\cdots) = \rho\left(1 - \frac{z^2}{c^2}\right).$$

Because $\rho = h(\cdots)$ and c depend upon z itself, the sign of F' is not always easy to determine. However, for a lot of interesting equations of state, F' vanishes only once, for a value z^* such that $z^* = c$. This happens for instance for a polytropic gas, because c^2 is an affine function of z.

We therefore place ourselves in the situation where F' vanishes only once. Then F is increasing for $z < z^*$ and decreasing for $z > z^*$. This yields an involution I such that $F \circ I = F$, which satisfies $I(z^*) = z^*$. Then z_+ must be equal to $I(u_- \cdot \nu)$. Finally,

$$u_+ = u_- + (z_+ - u_- \cdot \nu)\nu =: u_- + G(u_- \cdot \nu)\nu. \tag{A.8}$$

The Hugoniot locus is therefore a curve parametrized by the angle made by ν with the velocity u_-. Of course G depends also upon the parameter $\imath(\rho_-) = \kappa - \frac{1}{2}|u_-|^2$, but this is a constant in this analysis.

Algebraic aspects of the interaction. We apply formula (A.8) to the shocks R_1 and R_2. Let us denote ν_{12} (respectively ν_{21}) the unit normal pointing from U_1 (respectively from U_2) into U_{12}. This gives us two formulae for the unknown state u_{12}:

$$u_{12} = u_1 + G(u_1 \cdot \nu_{12})\nu_{12}, \quad u_{12} = u_2 + G(u_2 \cdot \nu_{21})\nu_{21}.$$

We therefore determine the unit normals ν_{12}, ν_{21} by solving the vectorial equation

$$u_1 + G(u_1 \cdot \nu_{12})\nu_{12} = u_2 + G(u_2 \cdot \nu_{21})\nu_{21}. \tag{A.9}$$

For moderate strength (u_2 and u_1 being close to each other), the interaction is close to that of the linearized system; Eq. (A.9) has a unique solution, where ν_{ij} is close to ν_j and u_{12} is close to $u_1 + u_2 - u_0$.

In order to solve practically (A.9), we can take the scalar product with ν_{12}, ν_{21}, u_1 and u_2. This gives

$$z_1 + G(z_1) = u_2 \cdot \nu_{12} + G(z_2)\nu_{21} \cdot \nu_{12},$$

$$z_2 + G(z_2) = u_1 \cdot \nu_{21} + G(z_1)\nu_{21} \cdot \nu_{12},$$

$$|u_1|^2 + G(z_1)z_1 = u_1 \cdot u_2 + G(z_2)u_1 \cdot \nu_{21},$$

$$|u_2|^2 + G(z_2)z_2 = u_1 \cdot u_2 + G(z_1)u_2 \cdot \nu_{12},$$

with $z_1 := u_1 \cdot \nu_{12}$ and z_2 accordingly. Together with the algebraic identity

$$z_1 z_2 - (u_1 \cdot \nu_{21})(u_2 \cdot \nu_{12}) = \sqrt{1 - (\nu_{12} \cdot \nu_{21})^2} \ \det(u_1, u_2),$$

this forms a system of five equations in five unknowns $(z_1, z_2, \nu_{12} \cdot \nu_{21}, u_1 \cdot \nu_{21}, \nu_{12} \cdot u_2)$.

In the limit case of zero strength, the figure is then symmetric and $u_{12} = u_1 = u_2$. We have $z_1 = z_2 = z^*$, meaning that R_1 and R_2 are sonic lines.

Binary interaction for the Chaplygin gas. The situation is somewhat different for a Chaplygin equation of state, because then the shocks are characteristic. Now F is a constant and Eq. (A.7) does not determine z_+. Instead, it reads

$$(u_- \cdot \nu)^2 - |u_-|^2 + 2\kappa = 0.$$

Because $\kappa = \frac{1}{2}(|u_-|^2 - c_-^2)$ in the present case, we obtain $u_- \cdot \nu = c_-$. This determines ν and thus the direction of the shock. Of course, u_- and u_+ play a symmetric role, and we therefore have $u_+ \cdot \nu = c_+$ as well, from which we deduce

$$u_+ = u_- + (c_+ - c_-)\nu. \tag{A.10}$$

We point out that across a Chaplygin shock, the Bernoulli $B := \frac{1}{2}(|u|^2 - c^2)$ is continuous, because $2B = |u \times \nu|^2 + ((u \cdot \nu)^2 - c^2) = |u \times \nu|^2$. In particular, the data $U_{1,2}$ are not independent of each other (as they used to be for a general gas), instead, they satisfy the compatibility condition $B_1 = B_0 = B_2$, that is

$$|u_1|^2 - c_1^2 = |u_2|^2 - c_2^2. \tag{A.11}$$

To determine u_{12}, we express it in two ways, by using (A.10) on both $R_{1,2}$. This gives us an equation

$$u_1 + (c_{12} - c_1)\nu_{12} = u_2 + (c_{12} - c_2)\nu_{21}. \tag{A.12}$$

Because ν_{12} and ν_{21} have already been determined (by $u_1 \cdot \nu_{12} = c_1$ and $u_2 \cdot \nu_{21} = c_2$), the unknown in (A.12) is the sound speed c_{12}. This vectorial equation, having a scalar unknown, seems overdetermined, but it turns out to admit a unique solution, because of (A.11). As a matter of fact, (A.6.1) can be used to define a velocity u_{12} by

$$u_{12} \times \nu_{12} = u_1 \times \nu_{12}, \quad u_{12} \times \nu_{21} = u_2 \times \nu_{21}.$$

We have thus $u_{12} = a_1 \nu_{12} + a_2 \nu_{21}$. We find that $a_j \nu_{12} \times \nu_{21} = \sqrt{2B_j}$ and therefore $a_1 = a_2$, from which we deduce $u_{12} \cdot \nu_{12} = u_{12} \cdot \nu_{21}$. This allows us to define ρ_{12} by the equality $c_{12} = u_{12} \cdot \nu_{12} = u_{12} \cdot \nu_{21}$. This completes the construction of U_{12}.

We recall that the pairwise interaction of Chaplygin shocks can be resolved by a nice geometrical procedure, see [9].

A.3. *The type of the steady-self-similar system*

As mentioned above, the stationary system is hyperbolic in the direction $\mathbf{e}$. However, the solutions we are searching are not only steady but also self-similar. This means that we have to solve the overdetermined (though compatible) system

$$\operatorname{div}_x F(U) = 0, \quad (x \cdot \nabla_x)U = 0. \tag{A.13}$$

The type of (A.13) at a state $\bar{U}$ and a point x is determined by the equation

$$P(\bar{U}; \xi) = 0, \quad x \cdot \xi = 0, \tag{A.14}$$

where

$$P(\bar{U}; \xi) = \det \frac{\partial F}{\partial U}(\bar{U}; \xi), \quad F(U; \xi) := \sum_j \xi_j F^j(U)$$

is the symbol of (1.5). The hyperbolicity of (1.5) is that $P(\bar{U}; \cdot)$ vanishes in some directions $\xi \in \mathbb{R}^3$. These directions forming a real projective curve $\operatorname{char}(\bar{U})$ on the unit sphere, whose intersection with planes containing the direction of hyperbolicity consists of m lines. When adding the constraint $(x \cdot \nabla_x)U = 0$, the system either remains hyperbolic, if $\operatorname{char}(\bar{U})$ meets the plane $x^\perp$, or becomes elliptic if $\operatorname{char}(\bar{U}) \cap x^\perp = \{0\}$. This is a striking example of the general rule that the type of a system of PDEs depends upon the relations we impose between the spacetime variables, and that an additional differential constraint is often a slip towards ellipticity.

In the context of the triple interaction problem, the system (A.13) is hyperbolic except in some conical neighborhood $\mathcal{K}$ of $\mathbf{e}$. Away from $\mathcal{K}$, we can use the unique continuation principle in influence domains to construct the solution explicitly. This is precisely what we do when we solve the primary and secondary interactions of planar waves.

The type of the linearized system about a state $\bar{U}$ changes across a cone $\mathcal{C} = \mathcal{C}(\bar{U})$, characteristic for the system (A.13). The word *characteristic* means that the normal ν to the cone at a point x satisfies

$$P(\bar{U}; \nu) = 0, \quad \text{and} \quad x \parallel \frac{\partial P}{\partial \xi}(\bar{U}; \nu). \tag{A.15}$$

Because P is homogeneous in ξ, this implies $x \cdot \nu = 0$. The characteristic cone is nothing but the dual of the cone defined by $P(\bar{U}; \xi) = 0$. When P is quadratic (likely if $m = 2$), then $\mathcal{C}(\bar{U})$ is a quadratic cone, that is a cone with circular basis. This happens for instance in gas dynamics.

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