

CONFLUENTES MATHEMATICI

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Tome 10, n° 2 (2018), p. 51-74.

http://cml.cedram.org/item?id=CML_2018__10_2_51_0

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REVERSIBLE PART OF QUANTUM DYNAMICAL SYSTEMS: A REVIEW

CARLO PANDISCIA

Abstract. In this work a quantum dynamical system $(\mathfrak{M}, \Phi, \varphi)$ is constituted by a von Neumann algebra \mathfrak{M} , a unital Schwartz map $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$ and a Φ -invariant normal faithful state φ on \mathfrak{M} . We will prove that the ergodic properties of a quantum dynamical system are determined by its reversible part $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$; i.e. by a von Neumann sub-algebra \mathfrak{D}_∞ of \mathfrak{M} , with an automorphism Φ_∞ and a normal state φ_∞ , as the restrictions on \mathfrak{D}_∞ . Moreover, if \mathfrak{D}_∞ is a trivial algebra, then the quantum dynamical system is ergodic. Furthermore, we will show some properties of reversible part of the quantum dynamical system, finally we will study its relations with the canonical decomposition of Nagy-Fojas of linear contraction related to a quantum dynamical system.

1. INTRODUCTION

This exploratory paper deals with the main properties of the reversible part of a discrete quantum dynamical system.

It is well-known that in the axiomatic approach of von Neumann-Segal-Mackey [10, 11], a physical system is characterized by a pair $(\mathfrak{A}, \mathfrak{S})$ constituted by a *-algebra with unit \mathfrak{A} and by a convex subset \mathfrak{S} of the dual space of algebra \mathfrak{A} .

According to Kadison [20], the temporal evolution of a dynamical system is described by an affine map $\Phi^\circ : \mathfrak{S} \rightarrow \mathfrak{S}$ which induces a positive map $\Phi : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $\omega(\Phi(a)) = \Phi^\circ(\omega)(a)$ for all $\omega \in \mathfrak{S}$ and $a \in \mathfrak{A}$. This passage from Φ° to Φ is referred to, in the physics literature, as the passage from the Heisenberg pictures to the Schrodinger pictures [11].

Physically the loss of pure states of system, during its temporal evolution is a reversibility index of the map Φ° .

Indeed, if the affine map Φ° is bijective (and therefore maps pure states into pure states), we obtain that Φ is a Jordan automorphism of the algebra of the observables [20]. in this case the physical system will be called reversible; however in the general case the loss of pure states during temporal evolution is unavoidable therefore the affine map Φ° is not bijective. Is it possible to outline a part of the observables algebra of physical system which is reversible along its temporary evolution? And, if this is not trivial, what are its main properties?

In this brief work we study the problem when the algebra of the observables is described by a von Neumann algebra which admits a normal stationary state for our temporal evolution.

We therefore consider a pair (\mathfrak{M}, Φ) composed of a von Neumann algebra \mathfrak{M} and a unital Schwartz map $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$, i.e. a normal map with $\Phi(1) = 1$ which satisfies the inequality:

$$0 \leq \Phi(a^*)\Phi(a) \leq \Phi(a^*a) , \quad a \in \mathfrak{M}. \quad (1.1)$$

The pair (\mathfrak{M}, Φ) is called (discrete) quantum process and Φ its dynamics.

Math. classification: 46L07, 46L55, 81S22.

Keywords: Quantum dynamical system, Multiplicative core, Algebra of effective observables.

A quantum process is called reversible if its dynamics is an automorphism of von Neumann algebras.

A quantum process $(\mathfrak{R}, \check{\Phi})$ is a sub-process of (\mathfrak{M}, Φ) if there is an injective homomorphism of von Neumann algebras $i : \mathfrak{R} \rightarrow \mathfrak{M}$ and a conditional expectation $\mathcal{E} : \mathfrak{M} \rightarrow i(\mathfrak{R})$ such that $\check{\Phi}^n = \mathcal{E} \circ \Phi^n \circ i$ for all natural numbers n .

The reversible part of a quantum process is its maximal reversible sub-process.

We denote with $\mathcal{B}(\mathcal{H})$ the C^* -algebra of the bounded linear operators on Hilbert space \mathcal{H} and with s and σ respectively, the ultrastrong operator topology and the ultraweakly operator topology on von Neumann algebra \mathfrak{M} , while with \mathfrak{M}_* its predual. Furthermore, a normal map or a normal state are σ -continuous maps [7].

We recall that a normal state φ on \mathfrak{M} is called either a stationary state for the quantum process (\mathfrak{M}, Φ) if $\varphi(\Phi(a)) = \varphi(a)$ for all $a \in \mathfrak{M}$ or an asymptotic equilibrium if $\Phi^n(a) \rightarrow \varphi(a)1$ as $n \rightarrow \infty$ in σ -topology.

Let (\mathfrak{M}, Φ) be a quantum process and φ its stationary state, the dynamics Φ admits a φ -adjoint if there is a normal unital Schwartz map $\Phi^\sharp : \mathfrak{M} \rightarrow \mathfrak{M}$ such that $\varphi(b\Phi(a)) = \varphi(\Phi^\sharp(b)a)$ for all $a, b \in \mathfrak{M}$.

The conditions for the existence of a φ -adjointness of dynamics of quantum process can be found in [1] and [28].

In what follow we will call a quantum dynamical system (QDS) a triple $(\mathfrak{M}, \Phi, \varphi)$ constituted by a quantum processes (\mathfrak{M}, Φ) and normal faithful stationary state φ where the dynamics Φ admits a φ -adjoint Φ^\sharp .

We have tried to keep the exposition as 'self-contained' as possible, recalling and proving some well known results of dynamical systems theory [14, 15] and of decoherence theory [4, 9].

This paper is organized as follows:

In section 2 we recall briefly some properties of multiplicative domain of Schwartz maps.

In section 3 we introduce the decomposition theorem for an algebraic probability space (\mathfrak{M}, φ) , constituted by a von Neumann algebra \mathfrak{M} and by one of its normal faithful states φ . Afterwards we study its connection with the multiplicative domain of dynamics of QDS.

In section 4 we study the connection between the decomposition theorem and the canonical decomposition of Nagy-Fojas of linear contraction on Hilbert space [27].

In section 5 we list some simple properties of the reversible part of a QDS in this case we prove that the ergodic properties of a QDS depend on the ergodic properties of its reversible part. Furthermore the decomposition th allows to define a new algebraic structure on \mathfrak{M} of $*$ -Banach algebra such that its dynamics Φ is a $*$ -homomorphism.

In section 6 by using some well known results on the Cesaro mean, we study under what conditions the QDS is completely irreversible, i.e. the reversible part is a trivial algebra.

2. MULTIPLICATIVE DOMAINS

The multiplicative domain \mathfrak{D}_Φ of a Schwartz map $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$, is defined as follows [30, 34]:

$$\mathfrak{D}_\Phi = \{a \in \mathfrak{M} : \Phi(a^*a) = \Phi(a^*)\Phi(a), \Phi(aa^*) = \Phi(a)\Phi(a^*)\} \quad (2.1)$$

and an element $a \in \mathfrak{D}_\Phi$ if and only if $\Phi(ax) = \Phi(a)\Phi(x)$ and $\Phi(xa) = \Phi(x)\Phi(a)$ for all $x \in \mathfrak{M}$.

It follows that \mathfrak{D}_Φ is a von Neumann algebra, since it is a unital *-algebra closed in the σ -topology, that is, \mathfrak{D}_Φ is the largest sub-algebra of \mathfrak{M} on which Φ behaves multiplicatively.

A consequence of Schwartz's inequality is the following remark:

If $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}$ is a unital Schwartz map which admits an inverse $\Phi^{-1} : \mathfrak{M} \rightarrow \mathfrak{M}$ (i.e. a unital Schwartz map such that $\Phi(\Phi^{-1}(a)) = \Phi^{-1}(\Phi(a)) = a$ for all $a \in \mathfrak{M}$), then Φ is an automorphism.

If \mathfrak{D}_∞^+ is the following von Neumann algebra:

$$\mathfrak{D}_\infty^+ = \bigcap_{n \in \mathbb{N}} \mathfrak{D}_{\Phi^n} \quad (2.2)$$

then we have that $\Phi(\mathfrak{D}_\infty^+) \subset \mathfrak{D}_\infty^+$ and Φ restricted to \mathfrak{D}_∞^+ is a *-homomorphism, but it is not surjective map.

We observe that

$$\mathfrak{D}_\infty^+ = \{a \in \mathfrak{D}_\Phi : \Phi^n(a) \in \mathfrak{D}_\Phi \text{ for all } n \in \mathbb{N}\}.$$

Defining the *multiplicative core* of Φ [33] as

$$\mathcal{C}_\Phi = \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+) \subset \mathfrak{D}_\infty^+ \quad (2.3)$$

we obtain that $\Phi(\mathcal{C}_\Phi) \subset \mathcal{C}_\Phi$.

Indeed for each $n \geq 0$ we have $\Phi^{n+1}(\mathfrak{D}_\infty^+) \subset \Phi^n(\mathfrak{D}_\infty^+)$ and

$$\Phi\left(\bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+)\right) \subset \bigcap_{n \in \mathbb{N}} \Phi(\Phi^n(\mathfrak{D}_\infty^+)) = \bigcap_{n \in \mathbb{N}} \Phi^{n+1}(\mathfrak{D}_\infty^+) = \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_\infty^+).$$

It is clear that the restriction of Φ to \mathcal{C}_Φ is a *-homomorphism.

Since Φ is a normal map and its restriction to \mathfrak{D}_∞^+ is a *-homomorphism, the set $\Phi^n(\mathfrak{D}_\infty^+)$ is a von Neumann algebra [7] therefore \mathcal{C}_Φ is a von Neumann algebra.

PROPOSITION 2.1. — *If Φ is an injective map on \mathfrak{D}_∞^+ then we obtain that $\Phi(\mathcal{C}_\Phi) = \mathcal{C}_\Phi$ and its restriction to the \mathcal{C}_Φ is a *-automorphism.*

Proof. — Let $y \in \mathcal{C}_\Phi$, by the definition of \mathcal{C}_Φ for each natural number $n \geq 0$ there exists an element $x_n \in \mathfrak{D}_\infty^+$ such that $y = \Phi^n(x_n)$ for all $n \geq 0$.

We have $\Phi(y) = \Phi^{n+1}(x_n) = \Phi^{n+1}(x_{n+1})$ for all $n \geq 0$ and by the injectivity of Φ we obtain that $x_n = x_{n+1}$ for all $n \in \mathbb{N}$ hence $y = \Phi(x_0)$. \square

Let φ be a stationary state for the quantum process (\mathfrak{M}, Φ) and $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ its GNS representation. It is well-known [28] that there is a unique linear contraction $U_{\Phi, \varphi}$ of $\mathfrak{B}(\mathcal{H}_\varphi)$ such that for each $a \in \mathfrak{A}$ we have:

$$U_{\Phi, \varphi} \pi_\varphi(a) \Omega_\varphi = \pi_\varphi(\Phi(a)) \Omega_\varphi. \quad (2.4)$$

Furthermore, if φ is a faithful state, then there is a unital Schwartz map $\Phi_\bullet : \pi_\varphi(\mathfrak{M}) \rightarrow \pi_\varphi(\mathfrak{M})$ such that

$$\Phi_\bullet(A) \Omega_\varphi = U_{\Phi, \varphi} A \Omega_\varphi, \quad A \in \pi_\varphi(\mathfrak{M}). \quad (2.5)$$

PROPOSITION 2.2. — *Let (\mathfrak{M}, Φ) be a quantum process and φ its faithful stationary state, we have:*

1. For each $d \in \mathfrak{D}_\Phi$ results $U_{\Phi, \varphi} \pi_\varphi(d) = \pi_\varphi(\Phi(d)) U_{\Phi, \varphi}$.

2. If $U_{\Phi, \varphi} \pi_{\varphi}(a) = \pi_{\varphi}(\Phi(a))U_{\Phi, \varphi}$ then for each $x \in \mathfrak{M}$ result such $\Phi(ax) = \Phi(a)\Phi(x)$.
3. $U_{\Phi, \varphi}^* U_{\Phi, \varphi} \in \pi_{\varphi}(\mathfrak{D}_{\Phi})'$ while $U_{\Phi, \varphi} U_{\Phi, \varphi}^* \in \pi_{\varphi}(\Phi(\mathfrak{D}_{\Phi}))'$.
4. The element d belongs to \mathfrak{D}_{Φ} if, and only if $\|U_{\Phi, \varphi} \pi_{\varphi}(d)\Omega_{\varphi}\| = \|\pi_{\varphi}(d)\Omega_{\varphi}\|$ and $\|U_{\Phi, \varphi} \pi_{\varphi}(d^*)\Omega_{\varphi}\| = \|\pi_{\varphi}(d^*)\Omega_{\varphi}\|$.

Proof. — It is straightforward. \square

By Schwartz's inequality and by the existence of a faithful stationary state for a quantum process (\mathfrak{M}, Φ) we obtain the following inclusions:

$$\cdots \mathfrak{D}_{\Phi^n} \subset \mathfrak{D}_{\Phi^{n-1}} \subset \cdots \mathfrak{D}_{\Phi^2} \subset \mathfrak{D}_{\Phi} \subset \mathfrak{M} \quad (2.6)$$

for all natural numbers $n \in \mathbb{N}$.

In this case, the map Φ is injective on \mathfrak{D}_{∞}^+ , since for each $a \in \mathfrak{D}_{\infty}^+$ with $\Phi(a) = 0$, we have:

$$\varphi(\Phi(a^*)\Phi(a)) = \varphi(\Phi(a^*a)) = \varphi(a^*a) = 0$$

hence its restricted to \mathcal{C}_{Φ} is a *-automorphism.

An example of a Schwartz map whose restriction to the multiplicative core \mathcal{C}_{Φ} is not an automorphism is described in Example 2.3.

Example 2.3. — Let \mathcal{H} be a Hilbert space and V an isometry of $\mathfrak{B}(\mathcal{H})$. We consider the unital Schwartz map:

$$\Phi(A) = V^*AV, \quad A \in \mathfrak{B}(\mathcal{H})$$

and we assume that there exists an element $\Omega \in \mathcal{H}$ such that $V\Omega = \Omega$.

For each natural number k we have

$$\mathfrak{D}_{\Phi^k} = \{E_k\}' , \quad E_k = V^k V^{*k}.$$

Indeed, if $A \in \{E_k\}'$ then A also belongs to \mathfrak{D}_{Φ^k} , since

$$\Phi^k(A^*A) = V^{*k}A^*AV^k = V^{*k}A^*AV^kV^{*k}V^k = V^*A^*V^kV^{*k}AV^k = \Phi(A^*)\Phi(A).$$

Let A belongs to \mathfrak{D}_{Φ^k} , we have:

$$V^{*k}AE_kB\Omega = V^{*k}AV^kV^{*k}B\Omega = \Phi^k(AB)\Omega = V^{*k}AB\Omega$$

It follows that for each $A \in \mathfrak{D}_{\Phi^k}$ we obtain:

$$V^{*k}AE_k = V^{*k}A \quad \implies \quad E_kAE_k = E_kA.$$

and hence

$$\mathfrak{D}_{\infty}^+ = \bigcap_{k \in \mathbb{N}} \{E_k\}'.$$

Let us notice that the algebra \mathfrak{D}_{∞}^+ is not trivial since one-rank projection $|\Omega\rangle\langle\Omega|$:

$$|\Omega\rangle\langle\Omega|\xi = \langle\Omega, \xi\rangle\Omega, \quad \xi \in \mathcal{H}$$

belongs to it.

Let $\Theta : \mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H})$ be any unital Schwartz map, we define

$$\Psi(A) = VAV^* + F_V\Theta(A)F_V, \quad A \in \mathfrak{B}(\mathcal{H})$$

where $F_V = I - VV^*$.

The unital Schwartz map Ψ has the following properties:

$$\Phi(\Psi(A)) = A, \quad A \in \mathfrak{B}(\mathcal{H})$$

and for each $A, B \in \mathfrak{B}(\mathcal{H})$

$$\langle \Omega, A\Phi(B)\Omega \rangle = \langle \Omega, \Psi(A)B\Omega \rangle.$$

Since $F_V E_k = 0$ for all $k \in \mathbb{N}$, we have:

$$\Psi(T) \in \{E_k\}' , \quad T \in \{E_k\}'.$$

Therefore the multiplicative core $\mathcal{C}_\Phi = \mathfrak{D}_\infty^+$.

3. DECOMPOSITION THEOREM

It is well known that any Hilbert space decomposes into direct sum of one its subspaces and its orthogonal complement. We have a similar result in von Neumann algebras theory.

Indeed, given a faithful normal state of a von Neumann algebra, for any its sub-algebra that satisfies a modular property we can decompose the von Neumann algebra in direct sum of vector spaces where a vector space is our sub-algebra and the other is its orthogonal related to the faithful state. We take as sub-algebra the multiplicative core of our unital Schwartz map consequently its orthogonal space is constituted by observables subject to dissipation. The temporal evolution is not a bijective map for these observables.

In this section we will study some properties of this algebraic decomposition and its relations with the orthogonal complement of Hilbert space of GNS representation obtained by our faithful normal state.

We consider a von Neumann algebra \mathfrak{M} and its faithful normal state φ and we denote by $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ the GNS representation of φ and denoting by $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ its modular automorphism group.

Let \mathfrak{A} be a von Neumann sub-algebra of \mathfrak{M} , we recall [22] that the φ -orthogonal of \mathfrak{A} is the set:

$$\mathfrak{A}^{\perp_\varphi} = \{a \in \mathfrak{M} : \varphi(a^*x) = 0 \text{ for all } x \in \mathfrak{A}\}. \quad (3.1)$$

The set $\mathfrak{A}^{\perp_\varphi}$ is a closed linear space in the σ -topology with $\mathfrak{A}^{\perp_\varphi} \cap \mathfrak{A} = \{0\}$.

We notice that $\mathfrak{A}^{\perp_\varphi} \subset \ker \varphi$ and if $\mathfrak{A} = \mathbb{C}I$ then $\mathfrak{A}^{\perp_\varphi} = \ker \varphi$, where $\ker \varphi = \{a \in \mathfrak{M} : \varphi(a) = 0\}$.

Moreover if $y \in \mathfrak{A}$ and $d_\perp \in \mathfrak{A}^{\perp_\varphi}$ then $yd_\perp \in \mathfrak{A}^{\perp_\varphi}$, since

$$\varphi((yd_\perp)^*x) = \varphi(d_\perp^*y^*x) = 0, \quad x \in \mathfrak{A}.$$

We notice since φ is a faithful state, the representation $\pi_\varphi : \mathfrak{M} \rightarrow \mathfrak{B}(\mathcal{H}_\varphi)$ is faithful normal map and $\pi_\varphi(\mathfrak{M})$ a von Neumann algebra isomorphic to \mathfrak{M} .

THEOREM 3.1. — *The von Neumann algebra \mathfrak{A} is invariant under modular automorphism group σ_t^φ if and only if both conditions are fulfilled:*

- a. *The set $\mathfrak{A}^{\perp_\varphi}$ is closed under the involution operation.*
- b. *For each $a \in \mathfrak{M}$ there is a unique $a_\parallel \in \mathfrak{A}$ and $a_\perp \in \mathfrak{A}^{\perp_\varphi}$ such that*

$$a = a_\parallel + a_\perp.$$

In other words we have the following algebraic decomposition:

$$\mathfrak{M} = \mathfrak{A} \oplus \mathfrak{A}^{\perp_\varphi}. \quad (3.2)$$

Proof. — From Takesaki [35] we have $\sigma_t^\varphi(\pi_\varphi(\mathfrak{A})) \subset \pi_\varphi(\mathfrak{A})$ for all $t \in \mathbb{R}$ if and only if there exists a normal conditional expectation $\mathcal{E} : \mathfrak{M} \rightarrow \mathfrak{A}$ such that $\varphi \circ \mathcal{E} = \varphi$.

Assuming \mathfrak{A} be invariant under modular automorphism group σ_t^φ is simple to prove that

$$\mathfrak{A}^{\perp\varphi} = \{a \in \mathfrak{M} : \mathcal{E}(a) = 0\}. \quad (3.3)$$

Therefore $\mathfrak{A}^{\perp\varphi}$ is closed under the involution operation.

For each $a \in \mathfrak{M}$ we set $a_\perp = a - \mathcal{E}(a)$ and

$$\varphi(a_\perp^* x) = \varphi((a^* - \mathcal{E}(a^*))x) = \varphi(a^* x) - \varphi(\mathcal{E}(a^*)x) = \varphi(a^* x) - \varphi(\mathcal{E}(a^* x)) = 0$$

for all $x \in \mathfrak{A}$, hence $a_\perp \in \mathfrak{A}^{\perp\varphi}$.

So for each $a \in \mathfrak{M}$ exists a unique $a_\parallel \in \mathfrak{A}$ and $a_\perp \in \mathfrak{A}^{\perp\varphi}$ such that $a = a_\parallel + a_\perp$ where we have set $a_\parallel = \mathcal{E}(a)$.

The uniqueness follows since if $a = 0$ then $a_\parallel = a_\perp = 0$.

Indeed, since $a_\parallel^* a_\perp$ and $a_\perp^* a_\parallel$ belongs to $\mathfrak{A}^{\perp\varphi}$, we have $\varphi(a_\parallel^* a_\perp) = \varphi(a_\perp^* a_\parallel) = 0$ and

$$\varphi(a^* a) = \varphi(a_\parallel^* a_\parallel) + \varphi(a_\perp^* a_\perp) = 0$$

with φ faithful state.

Vice-versa, if the set $\mathfrak{A}^{\perp\varphi}$ is closed under the involution operation and $\mathfrak{M} = \mathfrak{A} \oplus \mathfrak{A}^{\perp\varphi}$ then for each $a \in \mathfrak{M}$ exists a unique $a_\parallel \in \mathfrak{A}$ and $a_\perp \in \mathfrak{A}^{\perp\varphi}$ such that $a = a_\parallel + a_\perp$.

The map $a \in \mathfrak{M} \rightarrow a_\parallel \in \mathfrak{A}$ is a projection of norm one (i.e. it satisfies $(1)_\parallel = 1$ and $((a)_\parallel)_\parallel = a_\parallel$ for all $a \in \mathfrak{M}$) and for Tomiyama [36] it is a normal conditional expectation with $\varphi(a) = \varphi(a_\parallel)$ for all $a \in \mathfrak{M}$. \square

Remark 3.2. — If the set $\mathfrak{A}^{\perp\varphi}$ is a *-algebra (without unit) then $\mathfrak{A}^{\perp\varphi} = \{0\}$, since φ is a faithful state.

Moreover, if p is an orthogonal projector of \mathfrak{M} then $p \notin \mathfrak{A}^{\perp\varphi}$.

Remark 3.3. — If $a \in \mathfrak{M}$ with $a = a_\parallel + a_\perp$ where $a_\parallel \in \mathfrak{A}$ and $a_\perp \in \mathfrak{A}^{\perp\varphi}$, then

$$\|\pi_\varphi(a)\Omega_\varphi\|^2 = \|\pi_\varphi(a_\parallel)\Omega_\varphi\|^2 + \|\pi_\varphi(a_\perp)\Omega_\varphi\|^2. \quad (3.4)$$

PROPOSITION 3.4. — *Let \mathfrak{A} be a von Neumann algebra, invariant under modular automorphism group σ_t^φ . If \mathcal{H}_o and \mathcal{K}_o are respectively the closure of the linear space $\pi_\varphi(\mathfrak{A})\Omega_\varphi$ and $\pi_\varphi(\mathfrak{A}^{\perp\varphi})\Omega_\varphi$ then*

$$\mathcal{H}_\varphi = \mathcal{H}_o \oplus \mathcal{K}_o.$$

Moreover, the orthogonal projection P_o on Hilbert space \mathcal{H}_o belongs to $\pi_\varphi(\mathfrak{A})'$.

Proof. — We have that $\mathcal{K}_o \subset \mathcal{H}_o^\perp$, since for each $r_\perp \in \mathfrak{A}^{\perp\varphi}$ and $\psi_o \in \mathcal{H}_o$ then

$$\langle \pi_\varphi(r_\perp)\Omega_\varphi, \psi_o \rangle = \lim_{\alpha \rightarrow \infty} \langle \pi_\varphi(r_\perp)\Omega_\varphi, \pi_\varphi(r_\alpha)\Omega_\varphi \rangle = \lim_{\alpha \rightarrow \infty} \varphi(r_\perp^* r_\alpha) = 0$$

where $\psi_o = \lim_{\alpha \rightarrow \infty} \pi_\varphi(r_\alpha)\Omega_\varphi$ with $\{r_\alpha\}_\alpha$ net belongs to \mathfrak{A} .

If $\psi \in \mathcal{H}_\varphi$ we can write:

$$\psi = \lim_{\alpha \rightarrow \infty} \pi_\varphi(m_\alpha)\Omega_\varphi = \lim_{\alpha \rightarrow \infty} (\pi_\varphi(r_\alpha)\Omega_\varphi + \pi_\varphi((r_{\alpha\perp})\Omega_\varphi)$$

where $m_\alpha = r_\alpha + r_{\alpha\perp}$ for all α .

The net $\{\pi_\varphi(r_\alpha)\Omega_\varphi\}$ has limit, since of the relation (3.4) for each $\epsilon \geq 0$ there is a index ν such that for $\alpha \geq \nu$ and $\beta \geq \nu$, we have the Cauchy relation:

$$\|\pi_\varphi(r_\alpha)\Omega_\varphi - \pi_\varphi(r_\beta)\Omega_\varphi\| \leq \|\pi_\varphi(m_\alpha)\Omega_\varphi - \pi_\varphi(m_\beta)\Omega_\varphi\| \leq \epsilon$$

It follows that there are $\psi_{\parallel} \in \mathcal{H}_o$ and $\psi_{\perp} \in \mathcal{K}_o$ such that:

$$\psi = \lim_{\alpha \rightarrow \infty} \pi_{\varphi}(r_{\alpha})\Omega_{\varphi} + \lim_{\alpha \rightarrow \infty} \pi_{\varphi}((r_{\alpha\perp})\Omega_{\varphi} = \psi_{\parallel} + \psi_{\perp} \in \mathcal{H}_o \oplus \mathcal{K}_o .$$

It is simple to prove that $\pi_{\varphi}(\mathfrak{M})\mathcal{H}_o \subset \mathcal{H}_o$, so $P_o \in \pi_{\varphi}(\mathfrak{M})'$. \square

PROPOSITION 3.5. — *Let (\mathfrak{M}, Φ) be a quantum process and φ a normal faithful state on \mathfrak{M} . For each natural number n we have:*

$$\mathfrak{M} = \mathfrak{D}_{\Phi^n} \oplus \mathfrak{D}_{\Phi^n}^{\perp\varphi} \quad (3.5)$$

and

$$\mathfrak{M} = \mathfrak{D}_{\infty}^+ \oplus \mathfrak{D}_{\infty}^{+\perp\varphi} . \quad (3.6)$$

Furthermore, if φ is a stationary state for Φ then

$$\mathfrak{M} = \mathcal{C}_{\Phi} \oplus \mathcal{C}_{\Phi}^{\perp\varphi} \quad (3.7)$$

and the restriction of Φ to \mathcal{C}_{Φ} is a *-automorphism with $\Phi(\mathcal{C}_{\Phi}^{\perp\varphi}) \subset \mathcal{C}_{\Phi}^{\perp\varphi}$.

Proof. — For each $d \in \mathfrak{D}_{\Phi^n}$ and natural number n we get:

$$\Phi_{\bullet}^n(\sigma_t^{\varphi}(\pi_{\varphi}(d)^*)\sigma_t^{\varphi}(\pi_{\varphi}(d))) = \Phi_{\bullet}^n(\sigma_t^{\varphi}(\pi_{\varphi}(d)^*))\Phi_{\bullet}^n(\sigma_t^{\varphi}(\pi_{\varphi}(d)))$$

because Φ commutes with that modular automorphism group σ_t^{φ} .

It follows that $\sigma_t^{\varphi}(\pi_{\varphi}(\mathfrak{D}_{\Phi^n}))$ is into $\pi_{\varphi}(\mathfrak{D}_{\Phi^n})$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

If $b \in \mathcal{C}_{\Phi}$ then $\sigma_t^{\varphi}(\pi_{\varphi}(b)) \in \pi_{\varphi}(\mathcal{C}_{\Phi})$ for all real number t .

Indeed for each natural number n there is a $x_n \in \mathfrak{D}_{\infty}^+$ such that $b = \Phi^n(x_n)$, so we obtain:

$$\sigma_t^{\varphi}(\pi_{\varphi}(b)) = \sigma_t^{\varphi}(\pi_{\varphi}(\Phi^n(x_n))) = \Phi_{\bullet}^n(\sigma_t^{\varphi}(\pi_{\varphi}(x_n)))$$

and consequently $\sigma_t^{\varphi}(\pi_{\varphi}(x_n)) \in \pi_{\varphi}(\mathfrak{D}_{\infty}^+)$ for all natural number n .

Therefore $\sigma_t^{\varphi}(\pi_{\varphi}(b)) \in \pi_{\varphi}(\Phi^n(\mathfrak{D}_{\infty}^+))$ for all natural number n .

Let y belongs to $\mathcal{C}_{\Phi}^{\perp\varphi}$, since $\Phi(\mathcal{C}_{\Phi}) = \mathcal{C}_{\Phi}$, we have that for each $c \in \mathcal{C}_{\Phi}$

$$\varphi(\Phi(y)c) = \varphi(\Phi(y)\Phi(c_o)) = \varphi(yc_o) = 0$$

where $c = \Phi(c_o)$, with $c_o \in \mathcal{C}_{\Phi} \subset \mathfrak{D}_{\infty}^+$. \square

Now we deal with a QDS $(\mathfrak{M}, \Phi, \varphi)$ with φ -adjoint Φ^{\sharp} .

We define \mathfrak{D}_{∞} (or with $\mathfrak{D}_{\infty}(\Phi)$ when we have to highlight the map Φ) the following von Neumann algebra:

$$\mathfrak{D}_{\infty} = \bigcap_{k \in \mathbb{Z}} \mathfrak{D}_{\Phi^k} \quad (3.8)$$

where for each integer k we denote:

$$\Phi_k = \begin{cases} \Phi^k & k \geq 0 \\ \Phi^{\sharp|k|} & k < 0 \end{cases}$$

and \mathfrak{D}_{Φ^k} the von Neumann algebra of the multiplicative domain of the dynamics Φ^k .

Following Robinson [32] for each $a, b \in \mathfrak{M}$ and integer number k we define:

$$S_k(a, b) = \Phi_k(a^*b) - \Phi_k(a^*)\Phi_k(b) \in \mathfrak{M} \quad (3.9)$$

and we have these simple relations:

- a. $S_k(a, a) \geq 0$ for all $a \in \mathfrak{M}$ and integer k .
- b. $S_k(a, b)^* = S_k(b, a)$ for all $a, b \in \mathfrak{M}$ and integer k .

- c. $d \in \mathfrak{D}_\infty$ if and only if $S_k(d, d) = S_k(d^*, d^*) = 0$ for all integer k .
- d. $d \in \mathfrak{D}_\infty$ if and only if $\varphi(S_k(d, d)) = \varphi(S_k(d^*, d^*)) = 0$ for all integer k .
- e. The map $a, b \in \mathfrak{M} \rightarrow \varphi(S_k(a, b))$ for all integer k , is a sesquilinear form, hence

$$|\varphi(S_k(a, b))|^2 \leq \varphi(S_k(a, a))\varphi(S_k(b, b)), \quad a, b \in \mathfrak{M}.$$

Note that $\Phi(\mathfrak{D}_\infty) \subset \mathfrak{D}_\infty$ and $\Phi^\sharp(\mathfrak{D}_\infty) \subset \mathfrak{D}_\infty$.

Indeed, for each element $d \in \mathfrak{D}_\infty$ and integer k we get:

$$\varphi(S_k(\Phi(d), \Phi(d))) = \varphi(S_{k+1}(d, d)) = 0$$

and

$$\varphi(S_k(\Phi^\sharp(d), \Phi^\sharp(d))) = \varphi(S_{k-1}(d, d)) = 0.$$

Assuming that d^* belongs to \mathfrak{D}_∞ we have

$$\varphi(S_k(\Phi(d)^*, \Phi(d)^*)) = \varphi(S_k(\Phi^\sharp(d)^*, \Phi^\sharp(d)^*)) = 0.$$

It follows that restriction of the map Φ to the von Neumann algebra \mathfrak{D}_∞ is a *-automorphism with $\Phi(\Phi^\sharp(d)) = \Phi^\sharp(\Phi(d)) = d$ for all $d \in \mathfrak{D}_\infty$.

To sum up we have the following statement:

PROPOSITION 3.6. — *Let $(\mathfrak{M}, \Phi, \varphi)$ be a QDS. The restriction of Φ to \mathfrak{D}_∞ denoted with Φ_∞ is a *-automorphism of the von Neumann algebra \mathfrak{D}_∞ .*

*If \mathfrak{B} is a von Neumann sub-algebra of \mathfrak{M} such that the restriction of Φ to \mathfrak{B} is a *-automorphism then $\mathfrak{B} \subset \mathfrak{D}_\infty$.*

We have a (maximal) reversible QDS $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ where the normal state φ_∞ and the φ_∞ -adjoint Φ_∞^\sharp are respectively the restriction of φ and Φ^\sharp to the von Neumann algebra \mathfrak{D}_∞ .

Proof. — We prove that if the restriction of Φ to \mathfrak{B} is an automorphism then $\mathfrak{B} \subset \mathfrak{D}_\infty$.

As $\mathfrak{B} \subset \mathfrak{D}_{\Phi^n}$ for all natural number n and if $\Psi : \mathfrak{B} \rightarrow \mathfrak{B}$ is the map such that $\Psi(\Phi(b)) = \Phi(\Psi(b)) = b$ for all $b \in \mathfrak{B}$ then $\Psi(b) = \Phi^\sharp(b)$ since

$$\varphi(a\Psi(b)) = \varphi(\Phi(a\Psi(b))) = \varphi(\Phi(a)\Phi(\Psi(b))) = \varphi(\Phi(a)b) = \varphi(a\Phi^\sharp(b))$$

for all $a \in \mathfrak{M}$.

It follows that \mathfrak{B} is also Φ^\sharp -invariant, hence $\mathfrak{B} \subset \mathfrak{D}_{\Phi^n^\sharp}$ for all natural number n . \square

As it is clear that \mathfrak{D}_∞ is Φ_k -invariant for all integer number k and is invariant under automorphism group σ_t^φ , by the previous decomposition theorem follows:

PROPOSITION 3.7. — *Let $(\mathfrak{M}, \Phi, \varphi)$ be a QDS, there is a conditional expectation $\mathcal{E}_\infty : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$ such that*

- a. $\varphi \circ \mathcal{E}_\infty = \varphi$.
- b. $\mathfrak{D}_\infty^\perp = \ker \mathcal{E}_\infty$.
- c. $\mathfrak{M} = \mathfrak{D}_\infty \oplus \mathfrak{D}_\infty^\perp$.
- d. $\Phi_k(\mathfrak{D}_\infty^\perp) \subset \mathfrak{D}_\infty^\perp$ for all integer number k .
- e. $\mathcal{E}_\infty(\Phi_k(a)) = \Phi_k(\mathcal{E}_\infty(a))$ for all $a \in \mathfrak{M}$ and integer number k .
- f. $\mathcal{H}_\varphi = \mathcal{H}_\infty \oplus \mathcal{K}_\infty$ where \mathcal{H}_∞ and \mathcal{K}_∞ denotes the linear closure of $\pi_\varphi(\mathfrak{D}_\infty)\Omega_\varphi$ and of $\pi_\varphi(\mathfrak{D}_\infty^\perp)\Omega_\varphi$ respectively.

Proof. — The statements (a), (b) and (c) are simple consequences of theorem 3.1. For the statement (d), if $d_{\perp} \in \mathfrak{D}_{\infty}^{\perp\varphi}$ then for each integer number k and $x \in \mathfrak{D}_{\infty}$, we have $\varphi(\Phi_k(d_{\perp})^*x) = \varphi(d_{\perp}^*\Phi_{-k}(x)) = 0$ since $\Phi_{-k}(x) \in \mathfrak{D}_{\infty}$.

For the statement (e), for each $a, b \in \mathfrak{M}$ we obtain:

$$\begin{aligned} \varphi(b\mathcal{E}_{\infty}(\Phi_k(a))) &= \varphi((b_{\parallel} + b_{\perp})\mathcal{E}_{\infty}(\Phi_k(a))) = \varphi(b_{\parallel}\mathcal{E}_{\infty}(\Phi_k(a))) = \\ &= \varphi(\mathcal{E}_{\infty}(b_{\parallel}\Phi_k(a))) = \varphi(b_{\parallel}\Phi_k(a)) = \varphi(\Phi_{-k}(b_{\parallel})a) = \\ &= \varphi(\mathcal{E}_{\infty}(\Phi_{-k}(b_{\parallel})a)) = \varphi(\mathcal{E}_{\infty}(\Phi_{-k}(b_{\parallel})a)) = \\ &= \varphi(\Phi_{-k}(b_{\parallel})\mathcal{E}_{\infty}(a)) = \varphi(b_{\parallel}\Phi_k(\mathcal{E}_{\infty}(a))) = \\ &= \varphi((b_{\parallel} + b_{\perp})\Phi_k(\mathcal{E}_{\infty}(a))) = \varphi(b\Phi_k(\mathcal{E}_{\infty}(a))) \end{aligned}$$

where we have written $b = b_{\parallel} + b_{\perp}$ with $b_{\parallel} = \mathcal{E}_{\infty}(b)$. \square

The QDS $(\mathfrak{D}_{\infty}, \Phi_{\infty}, \varphi_{\infty})$ is called the reversible part of the QDS $(\mathfrak{M}, \Phi, \varphi)$.

Furthermore, a QDS is called *completely irreversible* if $\mathfrak{D}_{\infty} = \mathbb{C}1$.

In this case for each $a \in \mathfrak{M}$ we obtain $a = \varphi(a)1 + a_{\perp}$ and we can write:

$$\mathfrak{M} = \mathbb{C}1 \oplus \ker \varphi .$$

In the decoherence theory, the set \mathfrak{D}_{∞} is called *algebra of effective observables* of our QDS (see e.g. [4, 9]) and we highlight that the previous theorem is a particular case of a more general theorem that can be found in [5, 24].

We notice also that for all natural number n we obtain

$$\Phi^{\sharp n}(\Phi^n(d)) = d , \quad d \in \mathfrak{D}_{\infty}^+$$

and

$$\Phi^n(\mathfrak{D}_{\infty}^+) \subset \mathfrak{D}_{\Phi^{\sharp n}} .$$

We conclude this section with some simple remarks.

Remark 3.8. — The algebra of effective observables is independent by the stationary state φ since

$$\mathfrak{D}_{\infty} = \mathcal{C}_{\Phi} . \quad (3.10)$$

Indeed we have that $\mathfrak{D}_{\infty} \subset \bigcap_{n \in \mathbb{N}} \Phi^n(\mathfrak{D}_{\infty}^+)$ since $\mathfrak{D}_{\infty} \subset \mathfrak{D}_{\infty}^+$ and $\mathcal{C}_{\Phi} \subset \mathfrak{D}_{\infty}$ by theorem 3.6.

Summarizing, if the quantum process (\mathfrak{M}, Φ) admits a stationary state φ and the dynamic Φ admits a φ -adjoint then its reversible part is $(\mathcal{C}_{\Phi}, \Phi_{\infty})$.

Remark 3.9. — Let $(\mathfrak{M}, \Phi, \varphi)$ be a QDS, we denote with $\mathcal{A}(\mathcal{P})$ the von Neumann algebra generated by the set of all orthogonal projections $p \in \mathfrak{M}$ such that $\Phi_k(p) = \Phi_k(p)^2$ for all integer number k . It is easily proved that $\mathfrak{D}_{\infty} = \mathcal{A}(\mathcal{P})$ (see e.g [8]).

Remark 3.10. — If an orthogonal projection $P \in \mathfrak{M}$ satisfies the relation $\varphi(P) - \varphi(P)^2 = 0$ then $P \in \mathfrak{D}_{\infty}$.

Indeed by faithfulness of φ and by Schwartz inequality (1.1) we have $\Phi_k(P) - \Phi_k(P)^2 = 0$ for all integer number k . Therefore, if our QDS is completely irreversible then we have $\varphi(P) - \varphi(P)^2 > 0$ for all no trivial orthogonal projection P of \mathfrak{M} .

In section 6 we will find the conditions to have $\mathfrak{D}_{\infty} = \mathbb{C}1$ (see also [9] for the continuous case and $\mathfrak{D}_{\infty}^+ = \mathbb{C}1$).

4. DECOMPOSITION THEOREM AND LINEAR CONTRACTIONS

We are going to study the relations between canonical decomposition of Nagy-Fojas of linear contraction $U_{\Phi, \varphi}$ [27] and the decomposition theorem 3.7, therefore recalling the main statements of these topics.

A contraction T on the Hilbert space \mathcal{H} is called completely non-unitary (*c.n.u.*) if for no non zero, reducing subspace \mathcal{K} for T , is $T|_{\mathcal{K}}$ a unitary operator, where $T|_{\mathcal{K}}$ is the restriction of contraction T on the Hilbert space \mathcal{K} . We set with $D_T = \sqrt{I - T^*T}$ the defect operator of the contraction T and it is well-known that

$$TD_T = D_{T^*}T.$$

Moreover, $\|T\psi\| = \|\psi\|$ if, and only if $D_T\psi = 0$.

We consider the following Hilbert subspace of \mathcal{H} :

$$\mathcal{H}_0 = \{\psi \in \mathcal{H} : \|T^n\psi\| = \|\psi\| = \|T^{*n}\psi\| \text{ for all } n \in \mathbb{N}\}. \quad (4.1)$$

It is trivial to show that $T^n\mathcal{H}_0 = \mathcal{H}_0$ and $T^{*n}\mathcal{H}_0 = \mathcal{H}_0$ for all natural number n .

We then obtain the following canonical decomposition [27]:

THEOREM 4.1 (Sz-Nagy and Fojas). — *For every contraction T on \mathcal{H} there is a uniquely decomposition of \mathcal{H} into an orthogonal sum of two subspace reducing T ; we say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that $T_0 = T|_{\mathcal{H}_0}$ is unitary and $T_1 = T|_{\mathcal{H}_1}$ is c.n.u., where*

$$\mathcal{H}_0 = \bigcap_{k \in \mathbb{Z}} \ker(D_{T_k}) \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_0^\perp$$

with

$$T_k = \begin{cases} T^k & k \geq 0 \\ T^{*-k} & k < 0 \end{cases}.$$

We underline that the linear operators, $T_- = \text{so-} \lim_{n \rightarrow +\infty} T^{*n}T^n$ and $T_+ = \text{so-} \lim_{n \rightarrow +\infty} T^nT^{*n}$, exist in sense of strong operator (*so*) convergence (see [27] pag. 40).

We now focus our attention on the quantum dynamical systems $(\mathfrak{M}, \Phi, \varphi)$.

We set $V_- = \text{so-} \lim_{n \rightarrow +\infty} U_{\Phi, \varphi}^{*n} U_{\Phi, \varphi}^n$ and $V_+ = \text{so-} \lim_{n \rightarrow +\infty} U_{\Phi, \varphi}^n U_{\Phi, \varphi}^{*n}$, where $U_{\Phi, \varphi}$ that is the contraction as defined in (2.4).

It follows that for each $a, b \in \mathfrak{M}$, we obtain:

$$\lim_{n \rightarrow \pm\infty} \varphi(S_n(a, b)) = \langle \pi_\varphi(a)\Omega_\varphi, (I - V_\pm)\pi_\varphi(b)\Omega_\varphi \rangle \quad (4.2)$$

where $S_n(a, b)$ is given by (3.9).

We recall that, by the proposition 3.7 we get $\mathcal{H}_\varphi = \mathcal{H}_\infty \oplus \mathcal{K}_\infty$ with $U_k\mathcal{H}_\infty = \mathcal{H}_\infty$ and $U_k\mathcal{K}_\infty \subset \mathcal{K}_\infty$, where for every integer k

$$U_k = \begin{cases} U_{\Phi, \varphi}^k & k \geq 0 \\ U_{\Phi, \varphi}^{*-k} & k < 0 \end{cases}.$$

A simple consequence of proposition 2.2 is the following remark:

Remark 4.2. — For each integers k , we obtain that $a \in \mathfrak{D}_{\Phi_k}$ if and only if $\pi_\varphi(a)\Omega_\varphi \in \ker(D_{U_k})$ and $\pi_\varphi(a^*)\Omega_\varphi \in \ker(D_{U_k})$.

Therefore, $\mathcal{H}_\infty \subset \mathcal{H}_0$ because

$$\pi_\varphi(\mathcal{D}_\infty)\Omega_\varphi \subset \bigcap_{k \in \mathbb{Z}} \pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi \subset \bigcap_{k \in \mathbb{Z}} \ker(D_{U_k}).$$

We have shown that for each $a, b \in \mathfrak{M}$ and natural number k follows that (see [14] theorem 3.1):

$$\lim_{n \rightarrow +\infty} \varphi(S_k(\Phi^n(a), b)) = 0. \quad (4.3)$$

Indeed, for each natural numbers k and n

$$\varphi(S_k(\Phi^n(a), \Phi^n(b))) = \varphi(S_{k+n}(a, b)) - \varphi(S_n(a, b)),$$

and by the relation (4.2) results that $\lim_{n \rightarrow +\infty} (\varphi(S_{k+n}(a, b)) - \varphi(S_n(a, b))) = 0$.

Furthermore, for each natural number k and $a, b \in \mathfrak{M}$

$$|\varphi(S_k(\Phi^n(a), b))|^2 \leq \varphi(S_k(\Phi^n(a), \Phi^n(a)))\varphi(S_k(b, b)).$$

It follows that $\lim_{n \rightarrow +\infty} \varphi(S_k(\Phi^n(a), b)) = 0$.

We get a well-known statement [17, 24, 33]:

PROPOSITION 4.3. — *For all $a \in \mathfrak{M}$, every σ -limit point of the set $\{\Phi^k(a)\}_{k \in \mathbb{N}}$ belongs to the von Neumann algebra \mathfrak{D}_∞ .*

Moreover, for each d_\perp in $\mathfrak{D}_\infty^{\perp\varphi}$ we have:

$$\lim_{k \rightarrow +\infty} \Phi^k(d_\perp) = 0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} \Phi^{\sharp k}(d_\perp) = 0$$

where the limits are in σ -topology.

Proof. — If y is a σ -limit point of $\{\Phi^n(a)\}_{n \in \mathbb{N}}$, then there is a net $\{\Phi^{n_j}(a)\}_{j \in \mathbb{N}}$ such that $y = \lim_{j \rightarrow +\infty} \Phi^{n_j}(a)$ in σ -topology. Furthermore, for each $b \in \mathfrak{M}$

$$S_k(y, b) = \sigma - \lim_{j \rightarrow +\infty} [\Phi^k(\Phi^{n_j}(a)b) - \Phi^k(\Phi^{n_j}(a))\Phi^k(b)] = \sigma - \lim_{j \rightarrow +\infty} S_k(\Phi^{n_j}(a), b)$$

then from (4.3) we obtain $\lim_{j \rightarrow +\infty} \varphi(S_k(\Phi^{n_j}(a), b)) = 0$ hence $\varphi(S_k(y, b)) = 0$ and in the end follows that $\varphi(S_k(y, y)) = 0$ and $S_k(y, y) = 0$.

As that the adjoint is σ -continuous, then we have $y^* = \lim_{j \rightarrow +\infty} \Phi^{n_j}(a^*)$ and repeating the previous steps we find that $S_k(y^*, y^*) = 0$ hence $y \in \mathfrak{D}_\infty$.

Following to the last statement, we see that $\|\Phi^k(d_\perp)\| \leq \|d_\perp\|$ for all natural number k and because the unit ball of the von Neumann algebra \mathfrak{M} is σ -compact, we have proved that there is a subnet such that $\Phi^{k_\alpha}(d_\perp) \rightarrow y \in \mathfrak{D}_\infty^{\perp\varphi}$ in σ -topology. From the previous lemma, we can affirm that $y \in \mathfrak{D}_\infty \cap \mathfrak{D}_\infty^{\perp\varphi}$ as $y = 0$, then it can be only $\lim_{k \rightarrow +\infty} \Phi^k(d_\perp) = 0$ in σ -topology. \square

We have understood that the Hilbert space \mathcal{H}_∞ , which is the linear closure of $\pi_\varphi(\mathfrak{D}_\infty)\Omega_\varphi$, is contained in \mathcal{H}_0 . The next step is to understand when these two Hilbert spaces are equal.

Let $(\mathfrak{M}, \Phi, \varphi)$ be a QDS, so we can define for each integer k , the unital Schwartz map $\tau_k : \mathfrak{M} \rightarrow \mathfrak{M}$ as

$$\tau_k = \Phi_{-k} \circ \Phi_k, \quad k \in \mathbb{Z}. \quad (4.4)$$

For every integer k

1. $\varphi \circ \tau_k = \varphi$;
2. $\tau_k = \tau_k^\sharp$, where τ_k^\sharp is the φ -adjoint of τ_k .

Then for every integer k , we obtain the dynamical system $\{\mathfrak{M}, \tau_k, \varphi\}$ with

$$\mathfrak{D}_\infty(\tau_k) = \bigcap_{j \geq 0} \mathfrak{D}(\tau_k^j),$$

where $\mathfrak{D}(\tau_k^j)$ denotes the multiplicative domains of map τ_k^j .

From decomposition theorem 3.1, for every integer k

$$\mathfrak{M} = \mathfrak{D}_\infty(\tau_k) \oplus \mathfrak{D}_\infty(\tau_k)^{\perp\varphi}$$

and following the proposition 3.4 we obtain that

$$\mathcal{H}_\varphi = \mathcal{H}_{(k)} \oplus \mathcal{K}_{(k)}$$

where $\mathcal{H}_{(k)}$ and $\mathcal{K}_{(k)}$ are the Hilbert spaces, respectively the closure of the linear space $\pi_\varphi(\mathfrak{D}_\infty(\tau_k))\Omega_\varphi$ and of $\pi_\varphi(\mathfrak{D}_\infty(\tau_k))^{\perp\varphi}\Omega_\varphi$.

PROPOSITION 4.4. — *If $\overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi}$ denotes the closure of the linear space $\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi$, it follows that*

$$\mathcal{H}_0 = \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi},$$

where the \mathcal{H}_0 is the Hilbert space of Nagy decomposition of theorem 4.1.

Furthermore, for each $a \in \mathfrak{M}$, $\xi_0 \in \mathcal{H}_0$ and integer k

$$U_{\Phi, \varphi}^k \pi_\varphi(a) \xi_0 = \pi_\varphi(\Phi^k(a)) U_{\Phi, \varphi}^k \xi_0. \quad (4.5)$$

Proof. — We assume that $\mathfrak{D}(\tau_k) \subset \mathfrak{D}_{\Phi_k}$ for all integers k .

In fact, if $a \in \mathfrak{D}(\tau_k)$, then

$$\begin{aligned} \varphi(\Phi_k(a^*a)) &= \varphi(a^*a) = \varphi(\tau_k(a^*a)) = \varphi(\tau_k(a^*)\tau_k(a)) = \\ &= \varphi(\Phi_{-k}(\Phi_k(a)^*)\Phi_{-k}(\Phi_k(a))) \leq \\ &\leq \varphi(\Phi_{-k}(\Phi_k(a)^*\Phi_k(a))) = \varphi(\Phi_k(a^*)\Phi_k(a)) \leq \\ &\leq \varphi(\Phi_k(a^*a)). \end{aligned}$$

This proves that $\varphi(S_k(a, a)) = 0$ and, in the same way, $\varphi(S_k(a^*, a^*)) = 0$ for all integers k .

We have proved that

$$\mathfrak{D}_\infty(\tau_k) = \bigcap_{j \in \mathbb{N}} \mathfrak{D}_{\tau_k^j} \subset \mathfrak{D}_{\tau_k} \subset \mathfrak{D}_{\Phi_k}.$$

If $\xi_0 \in \mathcal{H}_0$, then for each k integer and natural number n

$$(U_{\Phi, \varphi}^{*k}, U_{\Phi, \varphi}^k)^n \xi_0 = \xi_0$$

and, for each $r_\perp \in \mathfrak{D}_\infty(\tau_k)^{\perp\varphi}$, we can write that

$$\langle \pi_\varphi(r_\perp)\Omega_\varphi, \xi_0 \rangle = \langle (U_{\Phi, \varphi}^{*k}, U_{\Phi, \varphi}^k)^n \pi_\varphi(r_\perp)\Omega_\varphi, \xi_0 \rangle = \langle \pi_\varphi(\tau_k^n(r_\perp))\Omega_\varphi, \xi_0 \rangle$$

and for each integer number k result:

$$\lim_{n \rightarrow +\infty} \langle \pi_\varphi(\tau_k^n(r_\perp))\Omega_\varphi, \xi_0 \rangle = 0$$

because $\tau_k^n(r_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in σ -topology.

It follows that $\mathcal{H}_0 \subset [\pi_\varphi(\mathfrak{D}_\infty(\tau_k)^{\perp\varphi})\Omega_\varphi]^\perp = [\mathcal{K}_k]^\perp$.

Therefore, for every integers k , we obtain:

$$\mathcal{H}_0 \subset \mathcal{H}_{(k)} \subset \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi} \implies \mathcal{H}_0 \subset \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi}.$$

Let assume $\xi_0 \in \bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi}$, for every integers k , we have a net $d_{\alpha,k} \in \mathfrak{D}_{\Phi_k}$ such that $\pi_\varphi(d_{\alpha,k})\Omega_\varphi \rightarrow \xi_0$ as $\alpha \rightarrow \infty$ and, for $k \geq 0$, we can obtain

$$\begin{aligned} U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \xi_0 &= U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \lim_{\alpha} \pi_\varphi(d_{\alpha,k})\Omega_\varphi = \lim_{\alpha} U_{\Phi,\varphi}^{*k} U_{\Phi,\varphi}^k \pi_\varphi(d_{\alpha,k})\Omega_\varphi = \\ &= \lim_{\alpha} \pi_\varphi(d_{\alpha,k})\Omega_\varphi = \xi_0. \end{aligned}$$

In the same way, for $k \geq 0$, we have got $U_{\Phi,\varphi}^k U_{\Phi,\varphi}^{*k} \xi_0 = \xi_0$.

It follows that

$$\bigcap_{k \in \mathbb{Z}} \overline{\pi_\varphi(\mathfrak{D}_{\Phi_k})\Omega_\varphi} \subset \mathcal{H}_0.$$

The relationship (4.5) is a straightforward. \square

We note that, for each $a \in \mathfrak{M}$ and $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$

$$\lim_{n \rightarrow \infty} \varphi(a^* \Phi_n(d_\perp) a) = 0 \quad (4.6)$$

because, for each $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$, we obtain $\Phi_n(d_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in σ -topology.

The polarization identity allows to say that

$$\lim_{n \rightarrow \infty} \varphi(a \Phi_n(d_\perp) b) = 0, \quad a, b \in \mathfrak{M}, \quad d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$$

so because U_φ is a contraction, it follows that for each $\xi \in \mathcal{H}_\varphi$ and $\psi \in \mathcal{K}_\infty$

$$\lim_{n \rightarrow \infty} \langle \xi, U_{\Phi,\varphi}^n \psi \rangle = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \langle \xi, U_{\Phi,\varphi}^{*n} \psi \rangle = 0. \quad (4.7)$$

The proposition 4.5 is a simple statement on the Hilbert spaces \mathcal{H}_∞ and \mathcal{H}_0 .

PROPOSITION 4.5. — *If $\Phi^n(d_\perp) \rightarrow 0$ [$\Phi^{*n}(d_\perp) \rightarrow 0$] as $n \rightarrow \infty$ in s -topology for all $d_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$; then $\mathcal{H}_\infty = \mathcal{H}_0$ and $V_+ = P_\infty$ [$V_- = P_\infty$].*

Proof. — We observe that, for every $\psi \in \mathcal{K}_\infty$, $\|U_{\Phi,\varphi}^n \psi\| \rightarrow 0$ as $n \rightarrow \infty$, because for each $k \in \mathbb{N}$ there is $d_k^\perp \in \mathfrak{D}_\infty^{\perp\varphi}$ such that $\|\psi - \pi_\varphi(d_k^\perp)\Omega_\varphi\| < 1/k$; as $U_{\Phi,\varphi}^n$ is a linear contraction, for all natural number n , we obtain:

$$\|U_{\Phi,\varphi}^n \psi\| < \frac{1}{k} + \varphi(\Phi^n(d_k^\perp)^* \Phi^n(d_k^\perp)).$$

If $\xi_0 \in \mathcal{H}_0$ then $\xi_0 = \xi_\parallel + \xi_\perp$ with $\xi_\parallel \in \mathcal{H}_\infty$ and $\xi_\perp \in \mathcal{K}_\infty$.

As $\xi_\perp = \xi_0 - \xi_\parallel \in \mathcal{H}_0$

$$\|\xi_\parallel\| + \|\xi_\perp\| = \|U_{\Phi,\varphi}^n \xi_0\| = \|U_{\Phi,\varphi}^n \xi_\parallel + U_{\Phi,\varphi}^n \xi_\perp\| = \|\xi_\parallel\| + \|U_{\Phi,\varphi}^n \xi_\perp\|$$

for all natural numbers n then follows that $\xi_\perp = 0$.

Moreover, for each $\xi \in \mathcal{H}_\varphi$, we obtain that $U_{\Phi,\varphi}^{*n} U_{\Phi,\varphi}^n \xi = \xi_0 + U_{\Phi,\varphi}^{*n} U_{\Phi,\varphi}^n \xi_1$ with $\xi_i \in \mathcal{H}_i$ for $i = 1, 2$ and $V_+ \xi = \xi_0$, because $\|U_{\Phi,\varphi}^n \xi_1\| \rightarrow 0$ as $n \rightarrow \infty$. \square

Remark 4.6. — We recall that a QDS $\{\mathfrak{M}, \Phi, \omega\}$ is mixing, if

$$\lim_{n \rightarrow \infty} \varphi(a \Phi^n(b)) = \varphi(a) \varphi(b), \quad a, b \in \mathfrak{M}. \quad (4.8)$$

By the relation (4.6), we can obtain that $\{\mathfrak{M}, \Phi, \omega\}$ is mixing if and only if its reversible part $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ is mixing.

Furthermore, let $\{\mathfrak{M}, \Phi, \omega\}$ be a mixing Abelian QDS, then there is a measurable dynamics space (X, \mathcal{A}, μ, T) such that \mathfrak{D}_∞ is isomorphic to the von Neumann algebra $L^\infty(X, \mathcal{A}, \mu)$ of the measurable bounded function on X . If the set X is a metric space and φ_∞ is the unique stationary state of \mathfrak{D}_∞ for the dynamics Φ_∞ , then by the corollary 4.3 of [13] follows that $\mathfrak{D}_\infty = \mathbb{C}1$.

5. SOME PROPERTIES OF REVERSIBLE PART OF A QDS

In this section we give some consequences of the previous propositions.

5.1. Ergodicity properties. We prove that the ergodic properties of a quantum dynamical system depends on its reversible part, determined from the algebra of the effective observables \mathfrak{D}_∞ .

Let assume a QDS $(\mathfrak{M}, \Phi, \varphi)$ with φ -adjoint Φ^\sharp ; it is ergodic if:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)] = 0, \quad a, b \in \mathfrak{M}, \quad (5.1)$$

while it is weakly mixing if:

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)| = 0, \quad a, b \in \mathfrak{M}. \quad (5.2)$$

We will use again the following notations $a_{\parallel} = \mathcal{E}_\infty(a)$ while $a_{\perp} = a - a_{\parallel}$ for all $a \in \mathfrak{M}$, where $\mathcal{E}_\infty : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$ is the conditional expectation of decomposition theorem 3.7.

We have the following:

PROPOSITION 5.1. — *The QDS $(\mathfrak{M}, \Phi, \varphi)$ is ergodic [weakly mixing] if and only if the reversible QDS $(\mathfrak{D}_\infty, \Phi_\infty, \varphi_\infty)$ is ergodic [weakly mixing].*

Proof. — For each $a, b \in \mathfrak{M}$ we have got:

$$\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b) = \varphi(a\Phi^k(b_{\parallel})) + \varphi(a\Phi^k(b_{\perp})) - \varphi(a_{\parallel})\varphi(b_{\parallel}).$$

Moreover, $\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varphi(a\Phi^k(b_{\perp})) = 0$, because by relation (4.6) for every $a \in \mathfrak{M}$ we have $\lim_{k \rightarrow \infty} \varphi(a\Phi^k(b_{\perp})) = 0$, hence

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)] = \\ & = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [\varphi(a\Phi^k(b_{\parallel})) - \varphi(a_{\parallel})\varphi(b_{\parallel})] \end{aligned}$$

with $\varphi(a\Phi^k(b_{\parallel})) = \varphi(a_{\parallel}\Phi^k(b_{\parallel})) + \varphi(a_{\perp}\Phi^k(b_{\parallel}))$ and $\varphi(a_{\perp}\Phi^k(b_{\parallel})) = 0$ so the element $a_{\perp}\Phi^k(b_{\parallel}) \in \mathfrak{D}_{\infty}^{\perp\varphi}$. It follows that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)] = \\ & = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel})] . \end{aligned}$$

For the weakly mixing properties, we obtain

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)| = \\ & = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(a_{\parallel}\Phi^k(b_{\parallel})) + \varphi(a_{\perp}\Phi^k(b_{\parallel})) + \varphi(a\Phi^k(b_{\perp})) - \varphi(a)\varphi(b)| = \\ & = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel}) + \varphi(a\Phi^k(b_{\perp}))| . \end{aligned}$$

Moreover

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi(a\Phi^k(b_{\perp}))| = 0, \quad a, b \in \mathfrak{M} .$$

If our QDS $(\mathfrak{M}, \Phi, \varphi)$ is weakly ergodic then

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N |\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel})| = 0 ,$$

therefore

$$\begin{aligned} & | |\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel})| - |\varphi(a\Phi^k(b_{\perp}))| | \leq \\ & \leq |\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)| , \end{aligned}$$

while if the reversible QDS $(\mathfrak{D}_{\infty}, \Phi_{\infty}, \varphi_{\infty})$ is weakly mixing, then our QDS is weakly mixing because

$$|\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)| \leq |\varphi_{\infty}(a_{\parallel}\Phi_{\infty}^k(b_{\parallel})) - \varphi_{\infty}(a_{\parallel})\varphi_{\infty}(b_{\parallel})| + |\varphi(a\Phi^k(b_{\perp}))| .$$

□

5.2. Particular *-Banach algebra. Let $(\mathfrak{M}, \Phi, \varphi)$ be a QDS and $\mathcal{E}_{\infty} : \mathfrak{M} \rightarrow \mathfrak{D}_{\infty}$ the map of proposition 3.7. We can define into the set \mathfrak{M} another frame of *-Banach algebra changing the product among elements of \mathfrak{M} as follows:

$$a \times b = a_{\parallel} b_{\parallel} + a_{\parallel} b_{\perp} + a_{\perp} b_{\parallel} , \quad a, b \in \mathfrak{M} , \quad (5.3)$$

where we have denoted with $a_{\parallel} = \mathcal{E}_{\infty}(a)$ and with $a_{\perp} = a - a_{\parallel}$ for all $a \in \mathfrak{M}$.

We want to stress once more that $a_{\parallel} b_{\perp}, a_{\perp} b_{\parallel} \in \mathfrak{D}_{\infty}^{\perp\varphi}$ since $\mathcal{E}_{\infty}(a_{\parallel} b_{\perp}) = a_{\parallel} \mathcal{E}_{\infty}(b_{\perp}) = 0$ and $\mathcal{E}_{\infty}(a_{\perp} b_{\parallel}) = \mathcal{E}_{\infty}(a_{\perp}) b_{\parallel} = 0$.

We then obtain that

$$a_{\perp} \times b_{\perp} = 0 .$$

The $(\mathfrak{M}, +, \times)$ is a Banach *-algebra with a unit, because for each $a, b \in \mathfrak{M}$ we get:

$$\|a \times b\| \leq \|a\| \|b\| .$$

We define \mathfrak{M}^b this Banach *-algebra.

However \mathfrak{M}^b it is not a C*-algebra as for each $d_\perp \in \mathfrak{D}_\infty^\perp$, $d_\perp \neq 0$ we have that its spectrum in \mathfrak{M}^b is $\sigma(d_\perp) \subset \{0\}$, while $\|d_\perp\| \neq 0$.

Another consequence is also that for each $a, b \in \mathfrak{M}$ results

$$\Phi(a \times b) = \Phi(a) \times \Phi(b) .$$

Consequently that $\Phi : \mathfrak{M}^b \rightarrow \mathfrak{M}^b$ is a *-homomorphism of Banach algebra.

For φ -adjoint Φ^\sharp we have:

$$\varphi(a \times \Phi(b)) = \varphi(a_\parallel \Phi(b_\parallel)) = \varphi(\Phi^\sharp(a_\parallel) b_\parallel) = \varphi(\Phi^\sharp(a) \times b)$$

with $\Phi^\sharp : \mathfrak{M}^b \rightarrow \mathfrak{M}^b$ *-homomorphism of Banach algebra.

Moreover as $\varphi(a^* \times a) = \varphi(a_\parallel^* a_\parallel)$, if $\varphi(a^* \times a) = 0$ then $a_\parallel = 0$ and then φ is not a faithful state on \mathfrak{M}^b .

It is easy to prove that for each $a, b \in \mathfrak{M}^b$ we have $\varphi(a^* \times b^* \times b \times a) = \varphi(a_\parallel^* b_\parallel^* b_\parallel a_\parallel)$ and then

$$\varphi(a^* \times b^* \times b \times a) \leq \|b\| \varphi(a^* \times a)$$

so we can make the GNS representation $(\mathcal{H}_\varphi^b, \pi_\varphi^b, \Omega_\varphi^b)$ of the state φ on Banach *-algebra \mathfrak{M}^b with the following properties [12]:

The representation $\pi_\varphi^b : \mathfrak{M}^b \rightarrow \mathfrak{B}(\mathcal{H}_\varphi^b)$ is a continuous map, i.e. $\|\pi_\varphi^b(a)\| \leq \|a\|$ for all $a \in \mathfrak{M}^b$, while Ω_φ^b is a cyclic vector for *-algebra $\pi_\varphi^b(\mathfrak{M}^b)$ and

$$\varphi(a) = \langle \Omega_\varphi^b, \pi_\varphi^b(a) \Omega_\varphi^b \rangle_b , \quad a \in \mathfrak{M}^b .$$

Furthermore we can find a unitary operator $U_\varphi^b : \mathcal{H}_\varphi^b \rightarrow \mathcal{H}_\varphi^b$ such that

$$\pi_\varphi^b(\Phi(a)) = U_\varphi^b \pi_\varphi^b(a) U_\varphi^{b*} , \quad a \in \mathfrak{M}^b ,$$

because Φ and Φ^\sharp are *-homomorphism of Banach algebra and

$$\begin{aligned} U_\varphi^b \pi_\varphi^b(a) \pi_\varphi^b(b) \Omega_\varphi^b &= \pi_\varphi^b(\Phi(a \times b)) \Omega_\varphi^b = \pi_\varphi^b(\Phi(a)) \pi_\varphi^b(\Phi(b)) \Omega_\varphi^b = \\ &= \pi_\varphi^b(\Phi(a)) U_\varphi^b \pi_\varphi^b(b) \Omega_\varphi^b . \end{aligned}$$

The linear map $Z : \mathcal{H}_\varphi^b \rightarrow \mathcal{H}_\varphi$ so defined $Z \pi_\varphi^b(a) \Omega_\varphi^b = \pi_\varphi(\mathcal{E}_\infty(a)) \Omega_\varphi$ for all $a \in \mathfrak{M}$ is an isometry with adjoint $Z^* \pi_\varphi(a) \Omega_\varphi = \pi_\varphi^b(\mathcal{E}_\infty(a)) \Omega_\varphi^b$ for all $a \in \mathfrak{M}^b$.

Furthermore, we have $ZU_\varphi^{bn} = ZU_{\Phi, \varphi}^n$ for all natural number n .

5.3. Abelian algebra of effective observables. We are going to prove that for every QDS $(\mathfrak{M}, \Phi, \varphi)$, there is an abelian algebra $\mathcal{A} \subset \mathfrak{D}_\infty$ that contains the center $Z(\mathfrak{D}_\infty)$ of \mathfrak{D}_∞ and with $\Phi(\mathcal{A}) \subset \mathcal{A}$.

The question of the existence of an abelian subalgebra which remains invariant under the action of a given quantum Markov semigroup, is widely debated in [2, 31].

We consider a discrete quantum process (\mathfrak{M}, Φ) with a Φ *-automorphism defining as $\mathfrak{P}(\mathfrak{M})$ the pure states of \mathfrak{M} .

It is well-known that if $\omega(a) = 0$ for all $\omega \in \mathfrak{P}(\mathfrak{M})$ then $a = 0$ (see e.g. [6]).

For each $\omega \in \mathfrak{P}(\mathfrak{M})$ with \mathfrak{D}_ω , we set the multiplicative domain of the unital completely positive map $a \in \mathfrak{M} \rightarrow \omega(a)I \in \mathfrak{M}$ then

$$\mathfrak{D}_\omega = \{a \in \mathfrak{M} : \omega(a^*a) = \omega(a^*)\omega(a) \text{ and } \omega(aa^*) = \omega(a)\omega(a^*)\}$$

is a von Neumann sub-algebra of \mathfrak{M} .

PROPOSITION 5.2. — *The von Neumann algebra*

$$\mathcal{A} = \bigcap \{\mathfrak{D}_\omega : \omega \in \mathfrak{P}(\mathfrak{M})\}$$

is an abelian algebra with $\Phi(\mathcal{A}) \subset \mathcal{A}$. Furthermore, for every stationary state φ of our quantum process (\mathfrak{M}, Φ) , there is a φ -invariant conditional expectation $\mathcal{E}_\varphi : \mathfrak{M} \rightarrow \mathcal{A}$ such that

$$\mathcal{E}_\varphi \circ \Phi = \Phi$$

Proof. — If $a, b \in \mathcal{A}$, for each pure state ω of \mathfrak{M} , we have $\omega(ab) = \omega(a)\omega(b) = \omega(ba)$, then $\omega(ab - ba) = 0$ which brings to $ab - ba = 0$.

The von Neumann algebra \mathcal{A} is Φ -invariant $\Phi(\mathcal{A}) \subset \mathcal{A}$ as $\omega \circ \Phi \in \mathfrak{P}(\mathfrak{M})$ for all $\omega \in \mathfrak{P}(\mathfrak{M})$ because Φ is a *-automorphism. Then for each $a \in \mathcal{A}$ we have:

$$\omega(\Phi(a^*)\Phi(a)) = \omega(\Phi(a^*a)) = \omega(\Phi(a^*))\omega(\Phi(a))$$

which proves $\Phi(a) \in \mathcal{A}$.

Let $\{\sigma_\varphi^t\}_{t \in \mathbb{R}}$ be a modular group related to the GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ of φ . Since the state φ is normal and faithful, we have $\pi_\varphi(\mathcal{A})'' = \pi_\varphi(\mathcal{A})$ and $\sigma_\varphi^t(\pi_\varphi(\mathcal{A})) \subset \pi_\varphi(\mathcal{A})$ for all $t \in \mathbb{R}$.

Since σ_φ^t is a *-automorphism, for each $a \in \mathcal{A}$

$$\omega(\sigma_\varphi^t(a^*)\sigma_\varphi^t(a)) = \omega(\sigma_\varphi^t(a^*a)) = \omega(\sigma_\varphi^t(a^*))\omega(\sigma_\varphi^t(a)), \quad \omega \in \mathfrak{P}(\mathfrak{M}),$$

so $\omega \circ \sigma_\varphi^t \in \mathfrak{P}(\mathfrak{M})$ for all real number t .

From Takesaki theorem [35], we obtain that there is a conditional expectation $\mathcal{E}_\varphi : \mathfrak{M} \rightarrow \mathcal{A}$ such that

$$\pi_\varphi(\mathcal{E}_\varphi(m)) = \nabla^* \pi_\varphi(m) \nabla, \quad m \in \mathfrak{M},$$

where $\nabla : \overline{\pi_\varphi(\mathcal{A})\Omega_\varphi} \rightarrow \mathcal{H}_\varphi$ is the embedding map (see also [1]).

We recall that every pure state is multiplicative on the center $Z(\mathfrak{M}) = \mathfrak{M} \cap \mathfrak{M}'$ of \mathfrak{M} (see [26]), so we have $Z(\mathfrak{M}) \subset \mathfrak{D}_\omega$ for all pure states ω and $\mathcal{A} = \mathfrak{M}$ in abelian case. \square

Having $(\mathfrak{M}, \Phi, \varphi)$ as a QDS with φ -adjoint Φ^\sharp , by the decomposition theorem, we have a *-automorphism $\Phi_\infty : \mathfrak{D}_\infty \rightarrow \mathfrak{D}_\infty$ with \mathfrak{D}_∞ von Neumann algebra, so by the previous proposition, we can say that there exists an abelian algebra $\mathcal{A} \subset \mathfrak{D}_\infty$ with $\Phi(\mathcal{A}) \subset \mathcal{A}$ getting the following commutative diagram:

$$\begin{array}{ccccc} & \mathfrak{M} & \xrightarrow{\Phi} & \mathfrak{M} & \\ i_\infty & \uparrow & & \downarrow & \mathcal{E}_\infty \\ & \mathfrak{D}_\infty & \xrightarrow{\Phi_\infty} & \mathfrak{D}_\infty & \\ i_o & \uparrow & & \downarrow & \mathcal{E}_\varphi \\ & \mathcal{A} & \xrightarrow{\Phi_o} & \mathcal{A} & \end{array}$$

where i_∞ and i_o are the embeddig of \mathfrak{D}_∞ and \mathcal{A} respectively, while Φ_∞ and Φ_o are the restriction of Φ to \mathfrak{D}_∞ and \mathcal{A} respectively.

We notice that if the von Neumann algebra \mathfrak{M} is abelian then $\mathcal{A} = \mathfrak{D}_\infty$.

5.4. Dilation properties. A reversible QDS $(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi})$ is called a dilation of the QDS $(\mathfrak{M}, \Phi, \varphi)$, if the quantum process (\mathfrak{M}, Φ) is a sub-process of $(\widehat{\mathfrak{M}}, \widehat{\Phi})$ with conditional expectation \mathcal{E} such that [22]:

$$\widehat{\varphi} = \varphi \circ \mathcal{E}. \quad (5.4)$$

We want to underline that the problem of establishing when a given QDS admits a reversible dilation is still largely open [16].

The view point of the dilation theory is the reverse of the open dynamical system theory (see [18]). Briefly, we have a little physical system, indicated by \mathcal{S} , which interacting with its environment, the reservoir \mathcal{R} . The composed system $\mathcal{S} + \mathcal{R}$ is considered to be isolated, so that its dynamics is a *-automorphism. The open dynamical system theory is interested in the reduced dynamics, i.e. a sub-process of our reversible process under assumption that the initial state $\widehat{\varphi}$ of $\mathcal{S} + \mathcal{R}$ satisfies the (5.4).

The following proposition shows the connection between the algebra of effective observable and the reversible dilation (see also [29]).

PROPOSITION 5.3. — *If $(\widehat{\mathfrak{M}}, \widehat{\Phi}, \widehat{\varphi})$ is a dilation of QDS $(\mathfrak{M}, \Phi, \varphi)$, then*

$$\widehat{\Phi}(i(a)) = i(\Phi(a)) \quad \text{if and only if} \quad a \in \mathfrak{D}_{\Phi}.$$

Proof. — Having $i(\Phi(a)^*) i(\Phi(a)) = \widehat{\Phi}(i(a)^*) \widehat{\Phi}(i(a))$ follows that

$$\Phi(a^*) \Phi(a) = \mathcal{E}(i(\Phi(a)^* \Phi(a))) = \mathcal{E}(\widehat{\Phi}(i(a)^* i(a))) = \widehat{\Phi}(i(a)^* i(a)).$$

For vice-versa, if $y = i(\Phi(a)) - \widehat{\Phi}(i(a))$, then we have

$$y^* y = i(\Phi(a)^* \Phi(a)) - \widehat{\Phi}(i(a)^* i(a)) - i(\Phi(a)^*) \widehat{\Phi}(i(a)) + \widehat{\Phi}(i(a)^* i(a))$$

because $a \in \mathfrak{D}_{\Phi}$. It follows that $\mathcal{E}(y^* y) = 0$ with \mathcal{E} faithful map, then $y = 0$. \square

Let $\mathfrak{M} = \mathfrak{D}_{\infty} \oplus \mathfrak{D}_{\infty}^{\perp \widehat{\varphi}}$ be the decomposition of theorem 3.1 of our QDS $(\mathfrak{M}, \Phi, \varphi)$ and $\mathcal{E}_{\infty} : \mathfrak{M} \rightarrow \mathfrak{D}_{\infty}$ the conditional expectation defined in proposition 3.7, we have the following remark.

Remark 5.4. — For each $a \in \mathfrak{M}$ and integer k , we have:

$$\widehat{\Phi}^k(i(\mathcal{E}_{\infty}(a))) = i(\Phi_k(\mathcal{E}_{\infty}(a))).$$

We observe that

$$X \in i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}} \quad \text{if and only if} \quad \mathcal{E}(X) \in \mathfrak{D}_{\infty}^{\perp \varphi}$$

therefore $i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}} = \{X \in \widehat{\mathfrak{M}} : \widehat{\varphi}(X^* i(d)) = 0 \quad \forall d \in \mathfrak{D}_{\infty}\}$ and $\widehat{\varphi}(X^* i(d)) = \varphi(\mathcal{E}(X^*) d)$ for all $d \in \mathfrak{D}_{\infty}$.

We can write the following algebraic decomposition of linear spaces:

$$\widehat{\mathfrak{M}} = i(\mathfrak{D}_{\infty}) \oplus i(\mathfrak{D}_{\infty})^{\perp \widehat{\varphi}},$$

and the unital completely positive map $\widehat{\mathcal{E}}_{\infty} = i \circ \mathcal{E}_{\infty} \circ \mathcal{E}$ is a conditional expectation from $\widehat{\mathfrak{M}}$ onto $i(\mathfrak{D}_{\infty})$.

6. DECOMPOSITION THEOREM AND CESARO MEAN

In this section, we are going to study the connection between the decomposition theorem 3.1 and some ergodic results briefly introduced here below.

The following proposition is well-known [21, 25]:

PROPOSITION 6.1. — *Let $\{\mathfrak{M}, \tau, \omega\}$ be a QDS. We consider the Cesaro mean:*

$$s_n = \frac{1}{n+1} \sum_{k=0}^n \tau^k.$$

Then, there is a ω -conditional expectation \mathcal{E} of \mathfrak{M} onto fixed point

$$\mathcal{F}(\tau) = \{a \in \mathfrak{M} : \tau(a) = a\}$$

such that:

$$\lim_{n \rightarrow 0} \|\phi \circ s_n - \phi \circ \mathcal{E}\| = 0, \quad \phi \in \mathfrak{M}_*.$$

A simple consequence of the proposition 6.1 is the following remark:

Remark 6.2. — $\{\mathfrak{M}, \tau, \omega\}$ is ergodic if, and only if $\mathcal{F}(\tau) = \mathbb{C}1$.

Assuming $(\mathfrak{M}, \Phi, \varphi)$ be a QDS and $\tau_k : \mathfrak{M} \rightarrow \mathfrak{M}$ the Schwartz map defined in (4.4) it follows

PROPOSITION 6.3. — *For each integer k , we obtain:*

$$\mathcal{F}(\tau_k) = \mathfrak{D}_{\Phi_k}.$$

Proof. — Without lost generality we set $k = 1$, then $\tau_1 = \Phi^\sharp \circ \Phi$.

If $x \in \mathcal{F}(\tau_1)$, we can write $\varphi(\Phi(x^*)\Phi(x)) = \varphi(x^*\tau_1(x)) = \varphi(x^*x) = \varphi(\Phi(x^*x))$, then $x \in \mathfrak{D}_\Phi$. The converse is proved similarly. \square

Now let us ask when \mathfrak{D}_∞ is trivial algebra (see also [9] proposition 15).

PROPOSITION 6.4. — *If $\mathfrak{D}_\infty = \mathbb{C}1$, then the normal state φ is of asymptotic equilibrium and the QDS $(\mathfrak{M}, \Phi, \varphi)$ is ergodic.*

Proof. — By decomposition theorem $\mathfrak{M} = \mathbb{C}1 \oplus \mathfrak{D}_\infty^{\perp\varphi}$ and for each $a \in \mathfrak{M}$, results that $a = \varphi(a)1 + a_\perp$ with $a_\perp \in \mathfrak{D}_\infty^{\perp\varphi}$. It follows that

$$\Phi^n(a) = \varphi(a)1 + \Phi^n(a_\perp)$$

and $\Phi^n(a_\perp) \rightarrow 0$ in σ -top. \square

A simple consequence of the previous propositions is

COROLLARY 6.5. — *If the QDS $\{\mathfrak{M}, \tau_k, \varphi\}$ is ergodic for some integer k , then $\mathfrak{D}_\infty = \mathbb{C}1$.*

Proof. — If we have ergodicity, then $\mathcal{F}(\tau_k) = \mathfrak{D}_{\Phi_k} = \mathbb{C}1$. \square

Summarizing:

$$\tau_1 \text{ ergodic} \implies \Phi \text{ completely irreversible} \implies \Phi \text{ ergodic}.$$

We can declare that, if $(\mathfrak{M}, \Phi, \varphi)$ is a QDS with Φ homomorphism, we obtain $\tau_1 = \Phi^\sharp \circ \Phi = id$. Hence while $(\mathfrak{M}, \Phi, \varphi)$ can be ergodic, the dynamical system $\{\mathfrak{M}, \tau_1, \varphi\}$ is not ergodic (if φ is not a multiplicative functional).

Example 6.6. — Let $M_n(\mathbb{C})$ be the complex n -dimensional matrix algebra and we consider the unital completely positive map $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ which fix diagonals.

It well known [23] that these maps have the form:

$$\Phi(X) = A * X + I * X, \quad X \in M_n(\mathbb{C})$$

where $*$ denote the Hadmard product, I denotes the identity matrix, A is a self-adjoint matrix with zero diagonal which satisfies $A + I \geq 0$.

Since $A * I = 0$, for each natural number n we have:

$$\Phi^n(X) = A^{(n)} * X + I * X, \quad X \in M_n(\mathbb{C})$$

with

$$A^{(n)} = \overbrace{A * A * \dots * A}^k.$$

We consider the following unital completely positive map Φ between $M_2(\mathbb{C})$:

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{1,1} & ax_{1,2} \\ ax_{2,1} & x_{2,2} \end{pmatrix}$$

where a is real number with $-1 \leq a \leq 1$ and φ the faithful state

$$\varphi(X) = tr(X) = x_{1,1} + x_{2,2}.$$

QDS $(M_2(\mathbb{C}), \Phi, \varphi(X))$ is not ergodic because for each $X, Y \in M_n(\mathbb{C})$ we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N [tr(X\Phi^k(Y)) - tr(X)tr(Y)] = -x_{1,1}y_{2,2} - x_{2,2}y_{1,1}.$$

The multiplicative domains of map Φ^n is given by

$$\mathfrak{D}_{\Phi^n} = \begin{pmatrix} x_{1,1} & 0 \\ 0 & x_{2,2} \end{pmatrix}, \quad n \in \mathbb{N}$$

which is an abelian algebra isomorphic to $\mathbb{C} \oplus \mathbb{C}$.

Furthermore, for each $X, Y \in M_n(\mathbb{C})$ results that

$$tr(X\Phi(Y)) = tr(\Phi(X)Y).$$

Therefore $\Phi^\sharp = \Phi$ and $\tau_k = \Phi^{2k}$ for all integers numbers k .

Summarizing we have that $\mathfrak{D}_\infty = \mathbb{C} \oplus \mathbb{C}$ and the automorphism $\Phi_\infty : \mathfrak{D}_\infty \rightarrow \mathfrak{D}_\infty$ is the identity map.

For each integer k , we consider the Schwartz map:

$$S_{n,k} = \frac{1}{n+1} \sum_{j=0}^n \tau_k^j. \quad (6.1)$$

From proposition 6.1 we have a positive map $\mathcal{E}_k : \mathfrak{M} \rightarrow \mathfrak{M}$ such that

$$\|\phi \circ S_{n,k} - \phi \circ \mathcal{E}_k\| \rightarrow 0, \quad \phi \in \mathfrak{M}_*$$

and \mathcal{E}_k is the conditional expectation related of von Neumann algebra \mathfrak{D}_{Φ^k} of theorem 3.1. Therefore $\mathcal{E}_k : \mathfrak{M} \rightarrow \mathfrak{D}_{\Phi^k}$ and $\varphi \circ \mathcal{E}_k = \varphi$, for all integers number k .

From relation 2.6 results that

$$\mathcal{E}_h \circ \mathcal{E}_k = \mathcal{E}_k, \quad k \geq h \geq 0$$

then follows $\mathfrak{D}_{\Phi^k} \subset \mathfrak{D}_{\Phi^h}$ for all $k \geq h$.

For each $a \in \mathfrak{M}$ result $\|\mathcal{E}_k(a)\| \leq \|a\|$ for all integers k and, if we apply the σ -compactness property for the bounded net $\{\mathcal{E}_k(a)\}_{k \in \mathbb{N}}$ of von Neumann algebra \mathfrak{M} , then we obtain that there is at least a σ -limit point $\mathcal{E}_+(a)$, therefore, there is a net $\{\mathcal{E}_{n_\alpha}(a)\}_\alpha$ such that $\mathcal{E}_+(a) = \sigma - \lim_\alpha \mathcal{E}_{n_\alpha}(a)$.

We obtain that $\mathcal{E}_+(a) \in \mathcal{D}_{\Phi^k}$ for all natural number k because, for each $a \in \mathfrak{M}$ follows $\mathcal{E}_h(\mathcal{E}_{n_\alpha}(a)) = \mathcal{E}_{n_\alpha}(a)$ when $n_\alpha \geq h$. Since \mathcal{E}_h are normal maps, it follows that

$$\mathcal{E}_h(\mathcal{E}_+(a)) = \mathcal{E}_+(a) , \quad h \in \mathbb{N}.$$

Furthermore, for each $x \in \mathfrak{D}_\infty^+$ we obtain that

$$\varphi(xa) = \lim_{\alpha \rightarrow \infty} \varphi(\mathcal{E}_{n_\alpha}(xa)) = \lim_{\alpha \rightarrow \infty} \varphi(x\mathcal{E}_{n_\alpha}(a)) = \varphi(x\mathcal{E}_+(a)) .$$

It follows that we get a unique σ -limit point $\mathcal{E}_+(a)$ for the net $\{\mathcal{E}_n(a)\}_{n \in \mathbb{N}}$.

Therefore, we obtain a map $\mathcal{E}_+ : \mathfrak{M} \rightarrow \mathfrak{D}_\infty^+$.

Moreover, $\mathcal{E}_{n_\alpha}(\mathcal{E}_+(a)) = \mathcal{E}_+(a)$ for all α , then $\mathcal{E}_+^2 = \mathcal{E}_+$ and for Tomiyama [36] the positive map \mathcal{E}_+ is a conditional expectation such that $\varphi \circ \mathcal{E}_+ = \varphi$; precisely it is the conditional expectation of relation (3.6).

We can also prove the following

PROPOSITION 6.7. — *Let $\{\mathfrak{M}, \Lambda_k, \varphi\}_{k \in \mathbb{N}}$ be a family of QDS. We consider the contraction $V_k : \mathcal{H}_\varphi \rightarrow \mathcal{H}_\varphi$, defined in (2.4), related to Schwartz map Λ_k :*

$$V_k \pi_\varphi(a) \Omega_\varphi = \pi_\varphi(\Lambda_k(a)) \Omega_\varphi , \quad a \in \mathfrak{M} .$$

If $\|[V_k^ - V_h^*] \xi\| \rightarrow 0$ as $h, k \rightarrow \infty$ for all $\xi \in \mathcal{H}_\varphi$, then there is a unital positive map $\Lambda : \mathfrak{M} \rightarrow \mathfrak{M}$, such that*

$$\|\phi \circ \Lambda_k - \phi \circ \Lambda\| \rightarrow 0 \tag{6.2}$$

as $k \rightarrow \infty$ for each $\phi \in \mathfrak{M}_$ with*

$$\varphi(\Lambda(a^*)\Lambda(a)) \leq \varphi(a^*a) , \quad a \in \mathfrak{M}$$

and $\varphi \circ \Lambda = \varphi$.

Proof. — Simple consequence of proposition 1.1 of [25] □

For each natural number n , we consider the following Schwartz map:

$$Z_n = \frac{1}{2n+1} \sum_{k=-n}^n \tau_k .$$

It is obvious that φ is a stationary state for Z_n with $\varphi(xZ_n(y)) = \varphi(Z_n(x)y)$ for all $x, y \in \mathfrak{M}$.

Moreover, for each $a \in \mathfrak{M}$ results

$$\begin{aligned} \pi_\varphi(Z_n(a)) \Omega_\varphi &= \frac{1}{2n+1} \sum_{k=-n}^n \pi_\varphi(\tau_k(a)) \Omega_\varphi = \\ &= \frac{1}{2n+1} \sum_{k=0}^n U_{\Phi, \varphi}^{*k} U_{\Phi, \varphi}^n \pi_\varphi(a) \Omega_\varphi + \\ &+ \frac{1}{2n+1} \sum_{k=1}^n U_{\Phi, \varphi}^k U_{\Phi, \varphi}^{*k} \pi_\varphi(a) \Omega_\varphi \end{aligned}$$

and because $U_{\Phi, \varphi}^{*n} U_{\Phi, \varphi}^n \rightarrow V_+$ and $U_{\Phi, \varphi}^n U_{\Phi, \varphi}^{*n} \rightarrow V_-$ in strong operator topology, we obtain:

$$\pi_\varphi(Z_n(a))\Omega_\varphi \rightarrow \frac{1}{2}(V_+ + V_-)\pi_\varphi(a)\Omega_\varphi$$

From the previous proposition follows that there is a φ invariant Schwartz map $Z : \mathfrak{M} \rightarrow \mathfrak{M}$ such that:

$$\|\phi \circ Z_n - \phi \circ Z\| \rightarrow 0, \quad \phi \in \mathfrak{M}_*$$

and

$$\pi_\varphi(Z(a))\Omega_\varphi = \frac{1}{2}(V_+ + V_-)\pi_\varphi(a)\Omega_\varphi.$$

We consider the decomposition $\mathfrak{M} = \mathfrak{D}_\infty \oplus \mathfrak{D}_\infty^\perp$, for each $a = a_\parallel + a_\perp \in \mathfrak{M}$ it results:

$$Z(a_\parallel + a_\perp) = a_\parallel + Z(a_\perp)$$

with $Z(a_\perp) \in \mathfrak{D}_\infty^\perp$.

We notice that, if $\Phi^n(d_\perp) \rightarrow 0$ and $\Phi^{\sharp n}(d_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in s -topology for all $d_\perp \in \mathfrak{D}_\infty^\perp$ (see proposition 4.5), $Z(d_\perp) = 0$ for all $d_\perp \in \mathfrak{D}_\infty^\perp$. We have a φ invariant Schwartz map $Z : \mathfrak{M} \rightarrow \mathfrak{D}_\infty$, such that

$$Z(xa) = xZ(a), \quad x \in \mathfrak{M}, a \in \mathfrak{D}_\infty.$$

It follows that Z is the conditional expectation \mathcal{E}_∞ of proposition 3.7.

We give an application of the previous results to quantum statistical inference theory, for further details, see [3, 19].

Let \mathfrak{M} be a von Neumann algebra and \mathcal{S} a family of normal states of \mathfrak{M}_* . The sub-algebra $\mathfrak{M}_o \subset \mathfrak{M}$ is called *sufficient* for $(\mathfrak{M}, \mathcal{S})$ if for each $a \in \mathfrak{M}$ exists $\hat{a} \in \mathfrak{M}_o$ such that $\omega(a) = \omega(\hat{a})$ for all $\omega \in \mathcal{S}$.

We have the following result (see also [19] theorem 3):

PROPOSITION 6.8. — *Let $(\mathfrak{M}, \Phi, \varphi)$ a quantum dynamical system, \mathcal{S} a family of normal states of \mathfrak{M}_* and $\tau_k : \mathfrak{M} \rightarrow \mathfrak{M}$ the Schwartz map defined in 4.4. If for $\omega \in \mathcal{S}$ we have:*

$$\omega \circ \tau_k = \omega, \quad k \in \mathbb{N} \tag{6.3}$$

then \mathfrak{D}_∞^+ is sufficient for $(\mathfrak{M}, \mathcal{S})$.

Proof. — By relation (6.3), we obtain that $\omega \circ \tau_k^j = \omega$ for all $j \in \mathbb{N}$. It follows that $\omega \circ S_{n,k} = \omega$ for all natural numbers j, k , where $S_{n,k}$ is the sum (6.1).

Since ω is normal state we have $\omega \circ \mathcal{E}_k = \omega$ for all $k \in \mathbb{N}$, where \mathcal{E}_k is the conditional expectation $\mathcal{E}_k : \mathfrak{M} \rightarrow \mathfrak{D}_{\Phi^k}$.

Therefore, we can say that $\omega(\mathcal{E}_+(a)) = \omega(a)$ for all $a \in \mathfrak{M}$ and \mathcal{E}_+ is the conditional expectation $\mathcal{E}_+ : \mathfrak{M} \rightarrow \mathfrak{D}_\infty^+$. \square

We notice that if the family \mathcal{S} is Φ and Φ^\sharp stationary, i.e. for every $\omega \in \mathcal{S}$

$$\omega \circ \Phi = \omega, \quad \omega \circ \Phi^\sharp = \omega$$

the condition 6.3 is fulfilled.

Moreover, with similar arguments of the previous proposition, if $\Phi^n(d_\perp) \rightarrow 0$ and $\Phi^{\sharp n}(d_\perp) \rightarrow 0$ as $n \rightarrow \infty$ in s -topology for all $d_\perp \in \mathfrak{D}_\infty^\perp$, then \mathfrak{D}_∞ is sufficient for $(\mathfrak{M}, \mathcal{S})$.

7. CONCLUSIONS

In this work, we studied the properties of the maximal reversible sub-system of a quantum dynamical system (QDS), called the reversible part. We proved that the ergodic properties of a QDS are induced by its reversible part and if it is trivial the QDS is ergodic.

We called a QDS completely irreversible when its reversible part is trivial, so that if the QDS is not ergodic, then it is not completely irreversible. Furthermore we given some conditions for the completely irreversibility.

The von Neumann algebra of our QDS has an algebraic decomposition in two linear spaces, in which one of them is constituted by the observable algebra of the reversible part (algebra of effective observables). We studied properties of this algebraic decomposition at the level of Hilbert spaces, and its relationships with the Nagy-Fojas decomposition for the linear contractions related to our QDS.

8. ACKNOWLEDGEMENTS

The author thanks R. Carbone, E. Sasso and V. Umanita for fruitful discussions during their hospitality at University of Study of Genova.

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Manuscript received September 25, 2017,
accepted July 26, 2018.

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