CONFLUENTES MATHEMATICI

THE MODEL THEORY OF COHEN RINGS

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Abstract. The aim of this article is to give a self-contained account of the algebra and model theory of Cohen rings, a natural generalization of Witt rings. Witt rings are only valuation rings in case the residue field is perfect, and Cohen rings arise as the Witt ring analogon over imperfect residue fields. Just as one studies truncated Witt rings to understand Witt rings, we study Cohen rings of positive characteristic as well as of characteristic zero. Our main results are a relative completeness and a relative model completeness result for Cohen rings, which imply the corresponding Ax-Kochen/Ershov type results for unramified henselian valued fields also in case the residue field is imperfect.

1. Introduction

The aim of this paper is to give an introduction to the model theory of complete Noetherian local rings A which have maximal ideal pA, where p is a prime number. From an algebraic point of view, the theory of such rings is classical. Under the additional hypothesis of regularity, they are valuation rings, and their study goes back to work of Krull ([9]) and many others. Structure theorems were obtained by Hasse and Schmidt ([5]), although there were deficiencies in the case that A/pA is not perfect. Further structural results were obtained by Witt ([27]) and Teichmüller ([23]). In particular Teichmüller gave a brief but precise account of the structure of such rings, even in the case that A/pA is imperfect. This was followed by Mac Lane ([16]), who improved upon Teichmüller's theory and proved relative structure theorems. Mac Lane built his work upon his study of Teichmüller's notion of p-independence in [22]. For further historical information, especially on this early period, the reader is encouraged to consult Roquette's article [18] on the history of valuation theory.

Turning away from the hypothesis of regularity, Cohen ([4]) gave an account of the structure of such rings. In fact his context was even more general: he did not assume Noetherianity.

Despite all of this work, more modern treatments (e.g. Serre, [20]) of this subject are often restricted to the case that A/pA is perfect. Consequently, the literature on the model theory of complete Noetherian local rings is sparse. For example, [25] also assumes that A/pA is perfect.

We became interested in the model theory of complete Noetherian local rings when we started to construct examples of NIP henselian valued fields with imperfect residue field in order to obtain an understanding of NIP henselian valued fields ([1]). After getting acquainted with the algebra of these rings as scattered in the literature detailed above, we realized that with a bit of tweaking, the proof ideas of these (classical) results can be used to gain an understanding of the model theory of such rings. To start with, this requires a careful recapitulation of the known algebraic (or structural) theory of such rings, bringing older results together in one

framework. This overview is given in Part I of the article. In this first part, many of the proof ideas are inspired by the work of others (and we point to the original sources), but we take care to prove everything which cannot be cited directly from elsewhere.

The underlying definition of a Cohen ring is the following:

DEFINITION 1.1 (Cf. Definitions 2.3 and 2.7). — A **Cohen ring** is a complete Noetherian local ring A with maximal ideal pA, where p is the residue characteristic of A.

A Cohen ring may either have characteristic 0 (in which case we call it strict) or p^n, where p is the characteristic of the residue field A/pA. In the second section, we introduce Cohen rings and recall that, for a given field k of positive characteristic, Cohen rings of every possible characteristic exist, which have residue field k. In the third section, we discuss and develop the machinery of multiplicative representatives, namely good sections of the residue map from the perfect core of the residue field into the Cohen ring A (Definition 3.1). We also comment on the extent to which these sections are unique, see Theorem 3.3, and how one can use them to generate Cohen rings (Proposition 3.6). In the fourth section, we prove that Teichmüller's embedding technique works in this context: we embed a Cohen ring with residue field k, with a choice of representatives, into the corresponding Cohen ring over the perfect hull of k (see Theorem 4.1). Building on this and using ideas from Cohen, we show that any two Cohen rings of the same characteristic and over the same residue field, both equipped with representatives, are isomorphic. In fact there is a unique isomorphism which respects the choices of representatives and is the identity on the residue field (Cohen Structure Theorem, 6.5). In the final section of the first part of the paper, we compare Cohen rings to Witt rings.

In the second part we begin a model-theoretic study, including describing the complete theories of Cohen ring of a fixed characteristic, over a given residue field. We work in the language $\mathfrak{L}_{\mathrm{vf}} = \mathfrak{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ of valued fields. We then show relative completeness, using a classical proof strategy together with the Cohen Structure Theorem from section 6. In particular, this result gives the following Ax-Kochen/Ershov principle:

THEOREM 1.2 (Cf. Corollary 8.5). — *Let (K, v) and (L, w) be two unramified henselian valued fields. Then*

$$\underbrace{Kv \equiv Lw}_{in\ \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{vK \equiv wL}_{in\ \mathfrak{L}_{\mathrm{oag}}} \iff \underbrace{(K, v) \equiv (L, w)}_{in\ \mathfrak{L}_{\mathrm{vf}}}.$$

Note that this was already claimed by Bélair in [2, Corollaire 5.2(1)]. However, since his proof crucially relies on Witt rings, it only works for perfect residue fields.

Moreover, we prove the following relative model-completeness result, which again essentially builds on the Cohen Structure Theorem.

THEOREM 1.3 (Cf. Corollary 9.4). — *Let $(K, v) \subseteq (L, w)$ be two unramified henselian valued fields. Then, we have*

$$\underbrace{Kv \preceq Lw}_{in\ \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{vK \preceq wL}_{in\ \mathfrak{L}_{\mathrm{oag}}} \iff \underbrace{(K, v) \preceq (L, w)}_{in\ \mathfrak{L}_{\mathrm{vf}}}.$$

In the penultimate section, we prove an embedding lemma (Proposition 10.1) similar to one proved by Kuhlmann for tame fields in [12]. This is the most delicate proof in the model-theoretic part of the paper. We then apply the embedding lemma to show relative existential closedness of unramified henselian valued fields assuming that the residue fields have a fixed finite degree of imperfection (Theorem 10.2).

Finally, applying the embedding lemma once again, we argue that in any unramified henselian valued field, there is no new structure induced on the residue field and value group:

THEOREM 1.4 (Cf. Theorem 11.3). — *Let (K, v) be an unramified henselian valued field. Then the value group vK and the residue field Kv are both stably embedded, as a pure ordered abelian group and as a pure field, respectively.*

We conclude by giving an example of a finitely ramified henselian valued field in which the residue field is no longer stably embedded (Example 11.5).

Part 1. **The structure of Cohen rings**

2. Pre-Cohen rings and Cohen rings

Throughout this paper, A, B, C will denote rings, which will always have a multiplicative identity 1 and be commutative; and k, l will be fields of characteristic p, which is a fixed prime number.

A ring A is **local** if it has a unique maximal ideal, which we will usually denote by $\mathfrak{m}$. A local ring is equipped with the **local topology**[1], which is the ring topology defined by declaring the descending sequence of ideals $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq ...$ to be a base of neighbourhoods of 0. The **residue field** of a local ring A, which we usually denote by k, is the quotient ring $A/\mathfrak{m}$, and the natural quotient map

$$\mathrm{res} : A \longrightarrow k$$

is called the **residue map**. The **residue characteristic** of A is by definition the characteristic of k.

For the sake of clarity, since maps between residue fields of local rings are of central importance in this paper, it will be suggestive to work with pairs (A, k) consisting of a local ring A together with its residue field k. Of course, such a pair is already determined by the local ring A, and this notation fails to explicitly mention the maximal ideal or the residue map. Without risk of confusion, we will also refer to such pairs as local rings.

LEMMA 2.1 (Krull, [10, Theorem 2]). — *Let A be a Noetherian local ring. Then $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \{0\}$. In other words, A is Hausdorff with respect to the local topology.*

Remark 2.2 (Other terminology). — Before we give our main definitions, namely Definitions 2.3 and 2.7, we note that many closely related ideas have been named in the literature, both in original papers and textbooks. Mac Lane, in [16], works with 'p-adic fields' and '$\mathfrak{p}$-adic fields'; whereas Cohen, in [4], prefers to work with 'local rings' (which, for Cohen, are necessarily Noetherian), 'generalized local rings', and 'v-rings'. Serre, in [20, Chapter II, §5], defines a 'p-ring' to be a ring A which

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[1]The local topology is also known as the $\mathfrak{m}$-*adic topology*.

is Hausdorff and complete in the topology defined by a decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset ...$ of ideals, such that $\mathfrak{a}_m \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$, and for which $A/\mathfrak{a}_1$ is a perfect ring of characteristic p. More recently, van den Dries, in [25, p. 132], defines a 'local p-ring' to be a complete local ring A with maximal ideal pA and perfect residue field A/pA.

To minimise the risk of confusion with existing terminology, we will not work with v-rings, p-adic fields, $\mathfrak{p}$-adic fields, p-rings, or local p-rings. Instead, since Warner's point of view, in [26, Chapter IX], is closer to our own, it is his definition of 'Cohen ring' that we adopt. We hope the reader will forgive us for this, but we feel that none of the other notions (several of which are arguably more standard in the literature) exactly captures the right context for this paper.

DEFINITION 2.3. — A **pre-Cohen ring** is a local ring (A, k) such that A is Noetherian and the maximal ideal $\mathfrak{m}$ is pA.

In particular, pre-Cohen rings are of residue characteristic p. Turning to the question of the characteristic of A itself, we note that a pre-Cohen ring need not even be an integral domain. However, a pre-Cohen ring is either of characteristic 0 or of characteristic p^m, for some $m \in \mathbb{N}_{>0}$.

LEMMA 2.4. — *For a pre-Cohen ring (A, k), the following are equivalent:*

   (i) *A is of characteristic zero,*

   (ii) *A is an integral domain,*

   (iii) *A is a valuation ring.*

In this case, the corresponding valuation v_A on the quotient field of A is of mixed characteristic $(0, p)$, has value group isomorphic to $\mathbb{Z}$, with $v_A(p)$ minimum positive, and has residue field k.

Proof. — This is a special case of [26, 21.4 Theorem].   □

DEFINITION 2.5. — If any (equivalently, all) of the conditions of Lemma 2.4 are satisfied, then we say that (A, k) is **strict**.

The word 'strict' is borrowed from Serre, [20, II,§5].

Remark 2.6. — In [4], Cohen writes in terms of regular Noetherian local rings. A local ring is **regular** if its Krull dimension is equal to the number of generators of its unique maximal ideal. In the case of a pre-Cohen ring (A, k), the maximal ideal is by definition generated by one element, namely p. Therefore, (A, k) is regular if and only if its Krull dimension is 1, which in turn holds if and only if (A, k) is strict.

A **morphism** of pre-Cohen rings, which we write as $\varphi : (A_1, k_1) \longrightarrow (A_2, k_2)$, is a pair $\varphi = (\varphi_A, \varphi_k)$ of ring homomorphisms $\varphi_A : A_1 \longrightarrow A_2$ and $\varphi_k : k_1 \longrightarrow k_2$, such that

   (i) $\mathfrak{m}_1 = \varphi_A^{-1}(\mathfrak{m}_2)$, i.e. φ_A is a morphism of local rings, and

   (ii) $\varphi_k \circ \mathrm{res} = \mathrm{res} \circ \varphi_A$.

This is nothing more than a way of speaking about morphisms of local rings as pairs of maps, to match the pairs (A, k). Every morphism φ_A of local rings induces a

ring homomorphism $\varphi_k : k_1 \longrightarrow k_2$ such that (φ_A, φ_k) is a morphism of pre-Cohen rings. From now on, by 'morphism' we mean a morphism of pre-Cohen rings. We will often (but not always) be concerned with morphisms $\varphi = (\varphi_A, \varphi_k)$ such that $k_2/\varphi_k(k_1)$ is separable. By an **embedding**, we mean a morphism $\varphi = (\varphi_A, \varphi_k)$ such that φ_A is injective. In the obvious way, we write $(A_1, k_1) \subseteq (A_2, k_2)$ if A_1 is a subring of A_2, k_1 is a subfield of k_2, and the inclusion maps form an embedding $(A_1, k_1) \longrightarrow (A_2, k_2)$.

DEFINITION 2.7 (Cf. [26, 21.3 Definition]). — A pre-Cohen ring (A, k) is a **Cohen** ring if it is also complete, i.e. complete with respect to the local topology.

Example 2.8. — $(\mathbb{Z}_p, \mathbb{F}_p)$ is a strict Cohen ring, and $(\mathbb{Z}_p/p^m\mathbb{Z}_p, \mathbb{F}_p)$ is a non-strict Cohen ring of characteristic p^m for each integer $m > 0$.

LEMMA 2.9. — *Every pre-Cohen ring of positive characteristic is already a Cohen ring.*

Proof. — In a non-strict pre-Cohen ring the topology is discrete. Thus it is complete. $\qquad\square$

Note that Cohen rings exist for any residue field and any characteristic. This foundational existence result goes back to the work of Hasse and Schmidt.

THEOREM 2.10 (Existence Theorem, [5, Theorem 20, p63]). — *Let k be a field of characteristic p. There exists a strict Cohen ring (A, k). Moreover, for each $m \in \mathbb{N}_{>0}$, there exists a Cohen ring (A_m, k) of characteristic p^m.*

Remark 2.11 (Inverse systems). — Let (A, k) be a non-strict Cohen ring of characteristic p^m, and let $n \in \mathbb{N}_{>0}$ such that $n \leqslant m$. The image of A under the quotient map $\mathrm{res}_{m,n} : A \longrightarrow A/\mathfrak{m}^n$ is a non-strict Cohen ring of characteristic p^n, again with residue field k. Similarly, if (A, k) is a strict Cohen ring, and $n \in \mathbb{N}_{>0}$, the image of A under the quotient map $r_n : A \longrightarrow A/\mathfrak{m}^n$ is a non-strict Cohen ring of characteristic p^n, and as before with residue field k. In both cases, the residue map of the quotient is induced by the residue map of A. These quotient maps are compatible in the sense that, for $n_1 \leqslant n_2 \leqslant m$, we have $r_{n_1} = \mathrm{res}_{n_2,n_1} \circ r_{n_2}$ and $\mathrm{res}_{m,n_1} = \mathrm{res}_{n_2,n_1} \circ \mathrm{res}_{m,n_2}$. In this way, given a strict Cohen ring (A, k), we obtain an inverse system of non-strict Cohen rings $(A_m, k)_{m>0}$ and maps $(\mathrm{res}_{m,n})_{n \leqslant m}$. Conversely, beginning with an inverse system of non-strict Cohen rings $(A_m, k)_{m>0}$, where each A_m has characteristic p^m, and quotient maps $(\mathrm{res}_{m,n})_{n \leqslant m}$, the inverse limit is a strict Cohen ring (A, k), with quotient maps $(r_m)_{m>0}$. Clearly, these constructions are mutually inverse.

Furthermore, if $\varphi : (A, k) \longrightarrow (B, l)$ is an embedding of Cohen rings that are both of characteristic p^m (resp., both strict), then φ induces embeddings $\varphi_n : (A/\mathfrak{m}^n, k) \longrightarrow (B/\mathfrak{m}^n, l)$, for all $n \leqslant m$ (resp., for all $n > 0$). On the other hand, if we begin with two inverse systems $(A_n, k)_{n>0}$ and $(B_n, l)_{n>0}$, such that $\mathrm{char}(A_n) = \mathrm{char}(B_n) = p^n$, and a compatible family of embeddings $\varphi_n : (A_n, k) \longrightarrow (B_n, l)$, for $n > 0$, there is a unique compatible embedding $\varphi : (A, k) \longrightarrow (B, l)$, where (A, k) and (B, l) are the corresponding inverse limits.

3. REPRESENTATIVES

3.1. **Teichmüller's multiplicative representatives.** The notion of 'representatives' plays a key role in this subject.

DEFINITION 3.1 (Cf. [23, §4.]). — Let (A, k) be a pre-Cohen ring, and let $\alpha \in k$. A **representative** of α is some $a \in A$ with $\mathrm{res}(a) = \alpha$. A **multiplicative representative** a of α is a representative which is also a p^n-th power in A, for all $n \in \mathbb{N}$. A **choice of representatives** is a partial function

$$s : k \dashrightarrow A$$

such that $s(\alpha)$ is a representative of α, for all α in the domain. To say that such a choice is **for** P means that P is the domain of s, i.e. $s : P \longrightarrow A$. Obviously, such a map is a **choice of multiplicative representatives** if $s(\alpha)$ is a multiplicative representative of α, for all α in the domain of s.

We observe that, for any pre-Cohen ring (A, k), there exist many choices of representatives for k, and of course for any subset P of k. It is obvious that the largest subfield of k for which multiplicative representatives may be chosen is the perfect core $k^{(p^\infty)}$, which is by definition the subfield of elements which are p^n-th powers, for all $n \in \mathbb{N}$. Note that $k^{(p^\infty)}$ is the largest perfect subfield of k. For any subset X of a ring, and any $n \in \mathbb{N}$, denote by $X^{(n)} = \{x^n \mid x \in X\}$ the set of n-th powers of elements of X.

The following straightforward lemma is the starting point for the study of multiplicative representatives.

LEMMA 3.2 ([24, Cf. Hilfssatz 8]). — *Let (A, k) be a pre-Cohen ring, let $a, b \in A$, and let $m, n \in \mathbb{N}$. If $a \equiv b \pmod{\mathfrak{m}^m}$, then $a^{p^n} \equiv b^{p^n} \pmod{\mathfrak{m}^{m+n}}$.*

Perhaps the most important result about multiplicative representatives is Theorem 3.3, which is also due to Teichmüller.

THEOREM 3.3 (Cf. [23, §4. Satz]). — *Let (A, k) be a Cohen ring. There exists a unique choice of multiplicative representatives for $k^{(p^\infty)}$:*

$$s : k^{(p^\infty)} \longrightarrow A.$$

The proof can be found in many places, for example [4, Lemma 7]. In fact, such a map s is also multiplicative in a stronger sense, namely that $s(\alpha)s(\beta) = s(\alpha\beta)$, for all $\alpha, \beta \in k^{(p^\infty)}$, cf. [20, Proposition 8(iii), §4, Ch. II].

LEMMA 3.4. — *Let (A, k) be a non-strict Cohen ring of characteristic p^{m+1}. There is a unique choice of representatives*

$$s : k^{(p^m)} \longrightarrow A^{(p^m)} \subseteq A.$$

Proof. — Let $\alpha \in k$, let $a, b \in A$, and suppose that both a^{p^m} and b^{p^m} are representatives of α^{p^m}. Then both a and b are representatives of α, and in particular $a \equiv b \pmod{\mathfrak{m}}$. By Lemma 3.2, $a^{p^m} \equiv b^{p^m} \pmod{\mathfrak{m}^{m+1}}$. Since the characteristic of A is p^{m+1}, we have $a^{p^m} = b^{p^m}$. □

The unique choice of representatives in Lemma 3.4 will be called the p^m-**representatives**.

3.2. λ-maps. A subset $\boldsymbol{\beta} \subseteq k$ is p-**independent** over a subfield $C \subseteq k$ if

$$[k^{(p)}C(\beta_1, ..., \beta_r) : k^{(p)}C] = p^r,$$

for all pairwise distinct elements $\beta_1, \ldots, \beta_r \in \boldsymbol{\beta}$, and for all $r \in \mathbb{N}$; and $\boldsymbol{\beta}$ is a p-**basis** over C if furthermore $k = k^{(p)}C(\boldsymbol{\beta})$. Equivalently, a p-basis is a maximal p-independent subset. If C is the prime field $\mathbb{F}_p$, then we just speak of 'p-independence' and 'p-bases'. The cardinality of a p-basis of k does not depend on the choice of any particular p-basis, and it is called the **imperfection degree**[2] of k. See [22], [14], and [15], for more information on p-independence and p-bases.

Our next task is to develop the theory of λ-maps and λ-representatives with respect to arbitrary p-independent subsets $\boldsymbol{\beta}$, which certainly may be infinite, since in general the imperfection degree of a field may be any cardinal number.

For a cardinal ν, and $n \in \mathbb{N}$, we define

$$P_{\nu,n} := \left\{ (i_\mu)_{\mu < \nu} \;\middle|\; |\{\mu < \nu \mid i_\mu \neq 0\}| < \infty \text{ and } \forall \mu < \nu,\, 0 \leqslant i_\mu < p^n \right\}$$

to be the set of the multi-indices of finite support, in ν-many elements, and in which each index is a non-negative integer strictly less that p^n. In this context, 'finite support' means that any such multi-index contains only finitely many non-zero indices. We emphasise that this set is just a technical device to facilitate our analysis of p-independence, and we will be mostly interested in the case $n = 1$. For a p-independent set $\boldsymbol{\beta} = (\beta_\mu)_{\mu < \nu}$, and $I = (i_\mu)_{\mu < \nu} \in P_{\nu,n}$, we write

$$\beta^I := \prod_{\mu < \nu} \beta_\mu^{i_\mu}$$

for the I-th monomial of $\boldsymbol{\beta}$. The set $\{\beta^I \mid I \in P_{\mu,n}\}$ is a $k^{(p^n)}$-linear basis of $k^{(p^n)}(\boldsymbol{\beta})$. Therefore, for each $\alpha \in k^{(p^n)}(\boldsymbol{\beta})$, there is a unique family $(\lambda_{\boldsymbol{\beta}}^I(\alpha))_{I \in P_{\nu,n}}$ of elements of k such that

$$\alpha = \sum_{I \in P_{\nu,n}} \beta^I \lambda_{\boldsymbol{\beta}}^I(\alpha)^{p^n}. \tag{3.2.1}$$

Note that this sum is finite since $\lambda_{\boldsymbol{\beta}}^I(\alpha)$ is zero for cofinitely many $I \in P_{\nu,n}$. We refer to

$$\lambda_{\boldsymbol{\beta}}^I : k^{(p^n)}(\boldsymbol{\beta}) \longrightarrow k$$
$$\alpha \longmapsto \lambda_{\boldsymbol{\beta}}^I(\alpha)$$

as the I-th λ-**map** with respect to $\boldsymbol{\beta}$.

3.3. **Representatives and subrings.** We prove in Proposition 3.6 that each Cohen ring of characteristic p^{m+1} is generated by the union of the p^m-representatives and a set of representatives for a p-basis of its residue field; this is a key ingredient of Theorem 5.1. We begin with a lemma.

LEMMA 3.5. — *Let (A, k) be a non-strict Cohen ring of characteristic p^{m+1}. Let $s : k \longrightarrow A$ be a choice of representatives. Then every element of A admits a unique representation as a sum $\sum_{i=0}^m s(\alpha_i)p^i$, where $\alpha_i \in k$. In particular, A is generated as a ring by the image $s(k)$ of s.*

Proof. — We claim that, for all $n \in \mathbb{N}$ and for all $a \in A$, there exist $\alpha_0, \ldots, \alpha_n \in k$ such that $a - \sum_{i=0}^n s(\alpha_i)p^i \in \mathfrak{m}_A^{n+1}$. We prove this by induction on n. The base case $n = 0$ follows from the fact that s is a choice of representatives for k. Suppose

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[2]Imperfection degree is sometimes called *Ershov degree* or *p-degree*.

the statement holds for some $n \in \mathbb{N}$. Let $a \in A$ and denote $\alpha := \mathrm{res}(a) \in k$. Let $\hat{a} \in A$ be such that $a = s(\alpha) + p\hat{a}$. By the inductive hypothesis, there exist $\alpha_0, \ldots, \alpha_n \in k$ such that $\hat{a} - \sum_{i=0}^{n} s(\alpha_i)p^i \in \mathfrak{m}^{n+1}$. Denote $b := s(\alpha) + \sum_{i=0}^{n} s(\alpha_i)p^{i+1}$. Now $a - b = p\hat{a} - p\sum_{i=0}^{n} s(\alpha_i)p^i \in \mathfrak{m}^{n+2}$.

The uniqueness part of the statement follows from the fact that multiplication by p^i is an isomorphism between the additive groups A/pA and $p^i A/p^{i+1}A$, for $i \leqslant m$. If the characteristic of A is p^{m+1}, then as soon as $m + 1 \leqslant n + 1$, the claim shows that A is generated by $s(k)$. $\qquad\square$

PROPOSITION 3.6. — *Let (A, k) be a non-strict Cohen ring characteristic p^{m+1}, and let $\boldsymbol{\beta}$ be a p-basis of k with representatives $s : \boldsymbol{\beta} \longrightarrow A$. Denote by $s_A : k^{(p^m)} \longrightarrow A^{(p^m)}$ the choice of p^m-representatives from Lemma 3.4. Then A is generated by $s(\boldsymbol{\beta}) \cup s_A(k^{(p^m)})$.*

Proof. — Let $B := [s(\boldsymbol{\beta}) \cup s_A(k^{(p^m)})]$ be the subring of A generated by the images of s and s_A. By Lemma 3.5, it suffices to show that B contains a set of representatives for each element of k. For $I = (i_\mu)_{\mu < \nu} \in P_{\nu,n}$, we write

$$s(\beta^I) := \prod_{\mu < \nu} (s(\beta_\mu))^{i_\mu}.$$

We now define a function $S : k \longrightarrow A$ by setting

$$S(\alpha) := \sum_{I \in P_{\nu,m}} s(\beta^I) s_A(\lambda_{\boldsymbol{\beta}}^I(\alpha)^{p^m}),$$

for $\alpha \in k$. Applying the residue map, and comparing with Equation (3.2.1), we can see that $S(\alpha)$ is a representative of α. Moreover, $S(\alpha)$ is an element of B, as required. $\qquad\square$

Remark 3.7 (Inverse systems with representatives). — Let k be a field of characteristic p. Let $(A_m, k)_{m \in \mathbb{N}}$ be an inverse system of Cohen rings, in the sense of Remark 2.11, where each A_m has characteristic p^{m+1}, and denote by $(\mathrm{res}_{n,m})_{n \geqslant m}$ the transition maps. Let $\boldsymbol{\beta}$ be a p-basis of k and, for each m, let $s_m : \boldsymbol{\beta} \longrightarrow A_m$ be a choice of representatives. Suppose that the maps s_m are compatible in the sense that $\mathrm{res}_{n,m} \circ s_n = s_m$, for $n \geqslant m$. Then $(A_m, k, s_m)_{m \in \mathbb{N}}$ forms an inverse system.

Let (A, k) be the inverse limit of $(A_m, k)_{m \in \mathbb{N}}$ with projections $(r_m)_{m \in \mathbb{N}}$. By Remark 2.11, (A, k) is a strict Cohen ring. The inverse limit of the maps s_m is a choice of representatives $s : \boldsymbol{\beta} \longrightarrow A$.

4. The Teichmüller Embedding Process

At the heart of all the structural arguments about Cohen rings is Teichmüller's embedding process, which we discuss in this section. The original formulation can be found in [23, §7]. Indeed, Mac Lane attributes this technique to Teichmüller, and describes it as the 'Teichmüller embedding process'. See [16, Theorem 6] for Mac Lane's version. In [4, Lemma 12], Cohen rewrote Teichmüller's embedding process for an arbitrary complete local ring.

THEOREM 4.1 (Teichmüller Embedding Process). — *Let (A, k) be a Cohen ring, let $\boldsymbol{\beta} \subseteq k$ be p-independent with representatives $s : \boldsymbol{\beta} \longrightarrow A$. There exists a Cohen ring $(A^T, k^T) \supseteq (A, k)$ such that*

(i) $k^T = k(\boldsymbol{\beta}^{(p^{-\infty})})$, where $\boldsymbol{\beta}^{(p^{-\infty})} = \{\beta^{p^{-n}} \mid \beta \in \boldsymbol{\beta}, n \in \mathbb{N}\}$,

(ii) s coincides with the restriction to $\boldsymbol{\beta}$ of the unique choice of multiplicative representatives $(k^T)^{(p^{\infty})} \longrightarrow A^T$.

Proof. — This proof is closely based on those of Teichmüller ([23, §7]) and Cohen ([4, Lemma 12]). It is a recursive construction. We begin by formally adjoining a p-th root of each $s(\beta)$, for each $\beta \in \boldsymbol{\beta}$. More constructively, we introduce a family of new variables $(X_\beta : \beta \in \boldsymbol{\beta})$, and let

$$A' := A[X_\beta : \beta \in \boldsymbol{\beta}]/\big(X_\beta^p - s(\beta) : \beta \in \boldsymbol{\beta}\big).$$

That is, A' is the quotient of the ring $A[X_\beta : \beta \in \boldsymbol{\beta}]$ by the ideal generated by the polynomials $X_\beta^p - s(\beta)$, for $\beta \in \boldsymbol{\beta}$. The natural map $A \longrightarrow A'$ is injective, and we identify A with its image in A'.

Taking the quotient of A' by pA' yields the field $k' := k(\beta^{p^{-1}} : \beta \in \boldsymbol{\beta})$, and so pA' is maximal. Indeed, since A is local with unique maximal ideal pA, the maximal ideals of A' are those lying over pA, which shows that pA' is the unique maximal ideal of A'. Thus (A', k') is a pre-Cohen ring, and we have $(A, k) \subseteq (A', k')$. Note that $\boldsymbol{\beta}^{(p^{-1})} := \{\beta^{p^{-1}} \mid \beta \in \boldsymbol{\beta}\}$ is p-independent in k'. Indeed, for each $\beta \in \boldsymbol{\beta}$, we write $s'(\beta^{p^{-1}})$ for the image of X_β in the quotient ring A'. Then $s' : \boldsymbol{\beta}^{(p^{-1})} \longrightarrow A'$ is a choice of representatives, and

$$s'(\beta^{p^{-1}})^p = s(\beta),$$

for all $\beta \in \boldsymbol{\beta}$.

Beginning with (A, k), we continue this process recursively, with recursive step $(A, k) \longmapsto (A', k')$. In this way, we construct a chain $(A_n, k_n)_{n \in \mathbb{N}}$ of pre-Cohen rings, such that $\boldsymbol{\beta}^{(p^{-n})} := \{\beta^{p^{-n}} \mid \beta \in \boldsymbol{\beta}\}$ is p-independent in $k_n = k(\boldsymbol{\beta}^{(p^{-n})})$ and $s_n : \boldsymbol{\beta}^{(p^{-n})} \longrightarrow A_n$ is a choice of representatives, such that

$$s_n(\beta^{p^{-n}})^{p^n} = s(\beta),$$

for all $n \in \mathbb{N}$ and all $\beta \in \boldsymbol{\beta}$.

The morphisms in this chain are embeddings, which we may even view as inclusions, by identifying each of (A_n, k_n) with its image in (A_{n+1}, k_{n+1}). The direct limit is a pre-Cohen ring $(A_\infty, k_\infty) \supseteq (A, k)$. Taking the completion, we obtain a Cohen ring $(A^T, k^T) \supseteq (A, k)$. The union $s^T := \bigcup_n s_n$ is a choice of representatives for $\boldsymbol{\beta}^T := \bigcup_n \boldsymbol{\beta}^{(p^{-n})}$ which commutes with the Frobenius map. By construction, we have $k^T = k(\boldsymbol{\beta}^{(p^{-\infty})})$, and so $\boldsymbol{\beta}^T \subseteq (k^T)^{(p^{\infty})}$. Therefore s^T coincides with the restriction to $\boldsymbol{\beta}^T$ of the unique choice of multiplicative representatives $(k^T)^{(p^{\infty})} \longrightarrow A^T$, as required. $\square$

5. Mac Lane's Identity Theorem

In this section we consider Cohen subrings of Cohen rings. We study the 'identity' of such subrings inside their overrings: in Theorem 5.1, which was first clearly articulated by Mac Lane, we show that such a subring is determined by a choice of representatives of a p-basis of its residue field.

Teichmüller's discussion of this issue can be found in [23, §8]. Developing these ideas, Mac Lane's theorems [16, Theorem 7] and [16, Theorem 12] show that a complete subfield of a $\mathfrak{p}$-adic field, in his language, is determined by a choice of

representatives for a p-basis of the residue field. Indeed, in our view, Mac Lane is the first to have clearly articulated this portion of the overall argument. Nevertheless, we closely follow Cohen's exposition, particularly relevant parts of his proof of [4, Theorem 11], which is in fact the theorem we will discuss in the next section.

THEOREM 5.1 (Mac Lane's Identity Theorem). — *Let (B, l) be a pre-Cohen ring, with pre-Cohen subrings (A_1, k_1) and (A_2, k_2). Suppose that (A_2, k_2) is a Cohen ring, i.e. is complete, and that $k_1 \subseteq k_2$. Let $\boldsymbol{\beta}$ be a p-basis of k_1 with representatives $s : \boldsymbol{\beta} \longrightarrow A_1$. If $s(\boldsymbol{\beta}) \subseteq A_2$ then $(A_1, k_1) \subseteq (A_2, k_2)$.*

Proof. — First we work in the case that (B, l) is non-strict of characteristic p^{m+1}. For $i \in \{1, 2\}$, let $s_i : k_i^{(p^m)} \longrightarrow A_i^{(p^m)} \subseteq A_i$, and let $s_B : l^{(p^m)} \longrightarrow B^{(p^m)} \subseteq B$, be the unique choices of p^m-representatives from Lemma 3.4. Let $\alpha \in k_i^{(p^m)}$. By Lemma 3.4, $s_B(\alpha)$ is the unique element of $B^{(p^m)}$ with residue α; but $s_i(\alpha)$ is another such element that happens to lie in $A_i^{(p^m)} \subseteq B^{(p^m)}$. Therefore $s_i(\alpha) = s_B(\alpha)$, which means that s_B extends s_i.

Since $k_1 \subseteq k_2 \subseteq l$, we have $k_1^{(p^m)} \subseteq k_2^{(p^m)} \subseteq l^{(p^m)}$. It follows that s_2 extends s_1. In particular, the image $s_1(k_1^{(p^m)})$ of s_1 is contained in the image of s_2, which in turn is contained in A_2. Also note that $s(\boldsymbol{\beta}) \subseteq A_2$, by assumption. By Proposition 3.6, the subring of B generated by $s_1(k_1^{(p^m)}) \cup s(\boldsymbol{\beta})$ is A_1. Therefore $A_1 \subseteq A_2$.

Finally, we suppose that (B, l) is strict. For all $m \in \mathbb{N}$, by the preceding paragraph, the residue ring of (A_1, k_1) of characteristic p^{m+1} is a subring of the corresponding residue ring of (A_2, k_2). Since (A_2, k_2) is complete by assumption, Remark 2.11 implies that (A_1, k_1) is a subring of (A_2, k_2). $\square$

6. COHEN'S HOMOMORPHISM THEOREM AND STRUCTURE THEOREM

The remaining ingredient of a structure theorem is the relationship between two arbitrary Cohen rings with the same residue field. Such a relationship exists, in the form of a morphism, and such a morphism is uniquely determined by specifying the image of a set of representatives of a p-basis of the residue field.

Cohen's paper [4] appears to be the first to study the case of characteristic p^m, $m > 0$. In this section we state and prove a version of Cohen's Theorem, [4, Theorem 11], suitable for our setting.

DEFINITION 6.1. — *Let (A, k) and (B, l) be pre-Cohen rings, and let $\varphi = (\varphi_A, \varphi_k) : (A, k) \longrightarrow (B, l)$ be a morphism. Also, let $\boldsymbol{\beta} \subseteq k$ be a p-basis of k, and let $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \varphi_k(\boldsymbol{\beta}) \longrightarrow B$ be representatives. We say that φ* **respects** *s_A and s_B if $\varphi_A \circ s_A = s_B \circ \varphi_k|_{\boldsymbol{\beta}}$.*

THEOREM 6.2 (Cohen's Homomorphism Theorem). — *Let (A, k) and (B, l) be Cohen rings, and let $\varphi_k : k \longrightarrow l$ be an embedding of fields such that $l/\varphi_k(k)$ is a separable extension. Let $\boldsymbol{\beta}$ be a p-basis of k and let $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \varphi_k(\boldsymbol{\beta}) \longrightarrow B$ be representatives. Suppose that (A, k) is strict or that (A, k) is non-strict of characteristic at least that of (B, l). Then there exists a unique ring homomorphism $\varphi_A : A \longrightarrow B$ such that*

$$\varphi = (\varphi_A, \varphi_k) : (A, k) \longrightarrow (B, l)$$
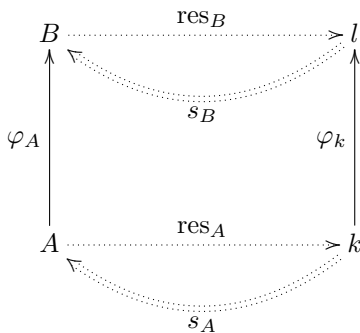
is a morphism which respects s_A and s_B.

FIGURE 6.1. Illustration of Definition 6.1

Moreover, if (A, k) and (B, l) have the same characteristic, then φ is an embedding.

Proof. — This proof is a construction closely based on that of Cohen ([4, Theorem 11]). For notational simplicity, we identify k with its image in l under the embedding φ_k. Then φ_k is the inclusion map id, and l/k is a separable extension. It suffices to construct a ring homomorphism $A \longrightarrow B$, which induces the inclusion map $k \longrightarrow l$. Throughout, we denote the residue maps by $\mathrm{res}_A : A \longrightarrow k$ and $\mathrm{res}_B : B \longrightarrow l$ respectively.

To begin with, we suppose that (A, k) is strict and that k is perfect. Thus $\boldsymbol{\beta}$ is empty, and we dispense with both of the maps s_A and s_B. Since (A, k) is a strict Cohen ring, we have $(\mathbb{Z}_p, \mathbb{F}_p) \subseteq (A, k)$, and there is the following natural ring homomorphism:

$$\varphi_0 : \mathbb{Z}_p \longrightarrow B.$$

Let T be a transcendence basis of $k/\mathbb{F}_p$. Since k is perfect, we have $T \subseteq k^{(p^\infty)} = k \subseteq l^{(p^\infty)} \subseteq l$. By Theorem 3.3, there are unique choices of multiplicative representatives: $s_{A,0} : k \longrightarrow A$ and $s_{B,0} : l^{(p^\infty)} \longrightarrow B$. Note that $s_{A,0}(T)$ is algebraically independent over $\mathbb{Z}_p$ since T is algebraically independent over $\mathbb{F}_p$. We consider the subring $\mathbb{Z}_p[s_{A,0}(T)] \subseteq A$, with $\mathrm{res}_A(\mathbb{Z}_p[s_{A,0}(T)]) = \mathbb{F}_p[T]$. We may extend φ_0 to a ring homomorphism

$$\varphi_{1,0} : \mathbb{Z}_p[s_{A,0}(T)] \longrightarrow B$$

by declaring $\varphi_{1,0}(s_{A,0}(t)) = s_{B,0}(t)$, for each $t \in T$. In fact, for each $n \in \mathbb{N}$, we consider the subring $\mathbb{Z}_p[s_{A,0}(T^{(p^{-n})})] \subseteq A$, with $\mathrm{res}_A(\mathbb{Z}_p[s_{A,0}(T^{(p^{-n})})]) = \mathbb{F}_p[T^{(p^{-n})}]$. As in the case $n = 0$, we construct a ring homomorphism

$$\varphi_{1,n} : \mathbb{Z}_p[s_{A,0}(T^{(p^{-n})})] \longrightarrow B,$$

by declaring $\varphi_{1,n}(s_{A,0}(t^{p^{-n}})) = s_{B,0}(t^{p^{-n}})$, for each $t \in T$. Since $s_{A,0}$ and $s_{B,0}$ are multiplicative, the family $(\varphi_{1,n})_{n \in \mathbb{N}}$ of ring homomorphisms forms an increasing chain. Taking the direct limit (i.e. union), we obtain the subring $A_0 := \mathbb{Z}_p[s_{A,0}(T^{(p^{-n})}) \mid n \in \mathbb{N}] \subseteq A$, with $\mathrm{res}_A(A_0) = \mathbb{F}_p[T^{(p^{-n})} \mid n \in \mathbb{N}]$, and we

obtain the ring homomorphism

$$\varphi_2 : A_0 \longrightarrow B.$$

Localising A_0 at $A_0 \cap pA$, we obtain the local ring $A_1 := (A_0)_{A_0 \cap pA} \subseteq A$, with $\mathrm{res}_A(A_1) = \mathbb{F}_p(T)^{\mathrm{perf}}$, and we extend φ_2 to a ring homomorphism

$$\varphi_3 : A_1 \longrightarrow B.$$

The final part of this construction is to extend φ_3 to have domain A. Since strict Cohen rings are henselian valuation rings, and $k/\mathbb{F}_p(T)^{\mathrm{perf}}$ is separable algebraic, this prolongation can be accomplished by a direct application of Hensel's Lemma, as in e.g. [11, Lemma 9.30]. More precisely, for a separable irreducible polynomial $f \in A_1[X]$ and $\alpha \in k$ with $\mathrm{res}_A(f)(\alpha) = 0$, by Hensel's Lemma we obtain $a \in A$ such that $f(a) = 0$. Likewise, we obtain $b \in B$ with $\varphi_3(f)(b) = 0$. We now extend φ_3 to a morphism

$$\varphi_4 : A_1[a] \longrightarrow B$$

by sending $a \longmapsto b$. Note that $\mathrm{res}_A(A_1[a]) = \mathbb{F}_p(T)^{\mathrm{perf}}(\alpha)$. Taking the direct limit of ring homomorphisms constructed in this way, we obtain a ring homomorphism

$$\varphi : A \longrightarrow B$$

that induces the inclusion map on the residue fields and respects s_A and s_B, as required. It remains to show that φ is the unique such morphism, but this follows from Theorem 5.1, applied to the case that $\boldsymbol{\beta}$ is empty.

Remaining under the assumption that (A, k) is strict, we turn to the case that k is imperfect. We are given a p-basis $\boldsymbol{\beta}$ of k with representatives $s_A : \boldsymbol{\beta} \longrightarrow A$ and $s_B : \boldsymbol{\beta} \longrightarrow B$. Note that $\boldsymbol{\beta}$ is p-independent in l, by our assumption that l/k is separable. By Theorem 4.1, there exists a Cohen ring $(A^T, k^T) \supseteq (A, k)$ such that

(i) $k^T = k^{\mathrm{perf}}$, and

(ii) s_A is the restriction to $\boldsymbol{\beta}$ of the multiplicative representatives

$$s_A^T : k^T \longrightarrow A^T.$$

By another application of Theorem 4.1, there exists a Cohen ring $(B^T, l^T) \supseteq (B, l)$ such that

(iii) $l^T = l(\boldsymbol{\beta}^{(p^{-\infty})})$, and

(iv) s_B is the restriction to $\boldsymbol{\beta}$ of the multiplicative representatives

$$s_B^T : (l^T)^{(p^\infty)} \longrightarrow B^T.$$

Since $k^T \subseteq l^T$ and k^T is perfect, by the first part of this proof there exists a unique morphism

$$\varphi : A^T \longrightarrow B^T$$

that induces the inclusion map on the residue fields.

By (ii), the composition $\varphi \circ s_A : \boldsymbol{\beta} \longrightarrow \varphi(A^T)$ coincides with the unique choice of multiplicative representatives for $\boldsymbol{\beta}$ in $\varphi(A^T)$; and by (iv), also s_B coincides with the unique choice of multiplicative representatives for $\boldsymbol{\beta}$ in B^T. Applying Theorem 3.3, and since $\varphi((A^T)^{(p^\infty)}) \subseteq (B^T)^{(p^\infty)}$, we have $\varphi \circ s_A = s_B$. If follows that both subrings $\varphi(A)$ and B of B^T contain $\varphi(s_A(\boldsymbol{\beta})) = s_B(\boldsymbol{\beta})$. Since also $k \subseteq l$, we may apply Theorem 5.1 to deduce that $\varphi(A) \subseteq B$. Therefore φ restricts to a ring

homomorphism $A \longrightarrow B$ that induces the inclusion map on the residue fields and respects s_A and s_B. The uniqueness of φ again follows from Theorem 5.1. Note that, the non-trivial proper ideals of A are $\mathfrak{m}_C^n$, for $n > 0$, and the quotient $A/\mathfrak{m}_C^n$ has characteristic p^n. Thus the morphism we have constructed is an embedding if and only if (B, l) is also strict. This completes the proof in the case that (A, k) is strict.

Finally, we turn to the case that (A, k) and (B, l) are non-strict, of characteristics p^n and p^m, respectively, for $n \geqslant m$. By Theorem 2.10, there exists a strict Cohen ring (C, k). Choose representatives $s_C : \boldsymbol{\beta} \longrightarrow C$. By the first part of this Theorem, there is a unique ring homomorphism $\varphi : C \longrightarrow A$ which induces the identity map on the residue field and respects s_C and s_A. Again note that the non-trivial proper ideals in (C, k) are $\mathfrak{m}_C^n$, for $n \in \mathbb{N}_{>0}$. Therefore φ is the composition of the quotient $C \longrightarrow C/\mathfrak{m}_C^n$ with an isomorphism $C/\mathfrak{m}_C^n \longrightarrow A$. Likewise there is a unique ring homomorphism $\psi : C \longrightarrow B$ which induces the inclusion map on the residue fields and respects s_C and s_B. Again ψ is the composition of the quotient of $C \longrightarrow C/\mathfrak{m}_C^m$ with an embedding $C/\mathfrak{m}_C^m \longrightarrow B$. Since $n \geqslant m$, $\mathfrak{m}_C^n \subseteq \mathfrak{m}_C^m$, and thus these homomorphisms give rise to a ring homomorphism $A \longrightarrow B$ that induces the inclusion map on the residue fields and respects s_A and s_B. Once again, the uniqueness follows from Theorem 5.1. Note that the morphism is an embedding if and only if $m = n$. $\square$

Remark 6.3. — In the setting of Theorem 6.2, and in the case that (A, k) is strict and (B, l) is of characteristic p^m, the resulting morphism φ factors into a composition of the natural quotient map $(A, k) \longrightarrow (A/\mathfrak{m}_A^m, k)$ and an embedding $(A/\mathfrak{m}_A^m, k) \longrightarrow (B, l)$.

In our applications of Cohen's Homomorphism Theorem in the second part of the paper, we require the following consequence:

COROLLARY 6.4 (Relative Embedding Theorem). — *Let (A_1, k_1) and (A_2, k_2) be two Cohen rings, and let (A_0, k_0) be a Cohen subring common to both. Assume we are given an embedding of residue fields $\varphi_k : k_1 \longrightarrow k_2$ over k_0 and that both k_1/k_0 and $k_2/\varphi_k(k_1)$ are separable. Then, there is an embedding φ of (A_1, k_1) into (A_2, k_2) which induces φ_k and fixes A_0 pointwise. Moreover, if φ_k is an isomorphism then φ is an isomorphism.*

Proof. — Note that by assumption, $(A_1, k_1), (A_2, k_2)$ and (A_0, k_0) have the same characteristic. Let $\boldsymbol{\beta}_0$ be a p-basis of k_0, and $s_0 : \boldsymbol{\beta}_0 \longrightarrow A_0$ be a choice of representatives. We first show the existence of an embedding of Cohen rings $\varphi : A_1 \longrightarrow A_2$ which induces φ_k and fixes $s_0(\boldsymbol{\beta}_0)$. Since k_1/k_0 is separable, we can find a p-basis $\boldsymbol{\beta}_1$ of k_1 prolonging $\boldsymbol{\beta}_0$, and a choice of representatives $s_1 : \boldsymbol{\beta}_1 \longrightarrow A_1$ prolonging s_0. Note that since φ_k restricts to the identity on k_0, $\varphi_k(\boldsymbol{\beta}_1)$ also contains $\boldsymbol{\beta}_0$. We now choose representatives $s_2 : \varphi_k(\boldsymbol{\beta}_1) \longrightarrow A_2$ such that s_2 prolongs s_0. By Theorem 6.2, and since $k_2/\varphi_k(k_1)$ is separable, there is a morphism $\varphi : A_1 \longrightarrow A_2$ that respects s_1 and s_2 (and hence fixes $s_0(\boldsymbol{\beta}_0)$) and induces φ_k. Since A_1 and A_2 have the same characteristic, φ is an embedding.

Now, let $\varphi : A_1 \longrightarrow A_2$ be any embedding which induces φ_k and fixes $s_0(\boldsymbol{\beta}_0)$. Then the restriction φ_0 of φ to A_0 is a ring isomorphism between A_0 and $\varphi_0(A_0)$. Since $s_0(\boldsymbol{\beta}_0)$ is contained in $\varphi_0(A_0)$, by Mac Lane's Identity Theorem (Theorem 5.1) we get $A_0 \subseteq \varphi_0(A_0)$. Symmetrically, as s_0 is also a choice of representatives

for a p-basis of the residue field of $\varphi_0(A_0)$, we get $\varphi_0(A_0) \subseteq A_0$. Therefore φ_0 is an automorphism of A_0. By assumption, φ_k restricts to the identity on k_0, and so φ induces the identity on the residue field of A_0. By construction of φ, φ_0 fixes $s_0(\boldsymbol{\beta}_0)$. Hence in particular φ_0 respects s_0 and s_0 (note that s_0 is a choice of representatives of the domain and codomain of φ_0, which are both A_0). Theorem 6.2 implies that there is a unique automorphism of A_0 with these properties. As the identity map from A_0 to A_0 also induces the identity on k_0 and fixes $s_0(\boldsymbol{\beta}_0)$, we conclude $\varphi_0 = \mathrm{id}_{A_0}$.

Finally, we show that if φ_k is an isomorphism, then φ is an isomorphism: if φ_k is an isomorphism and $\boldsymbol{\beta}_1$ is a p-basis of k_1, then $\varphi_k(\boldsymbol{\beta}_1)$ is a p-basis of k_2. Thus, $\varphi(A_1)$ contains the lift of a p-basis for k_2, and hence we have $A_2 \subseteq \varphi(A_1)$ by Mac Lane's Identity Theorem (Theorem 5.1). $\qquad\square$

More explicitly, given an isomorphism between the residue fields of two Cohen rings of the same characteristic, we get a complete understanding of its lifts to isomorphisms of Cohen rings:

COROLLARY 6.5 (Cohen Structure Theorem, v.1). — *Let (A_1, k_1) and (A_2, k_2) be two Cohen rings of the same characteristic, let $\varphi_k : k_1 \longrightarrow k_2$ be an isomorphism of residue fields, and let $\boldsymbol{\beta} \subseteq k_1$ be a p-basis. Consider representatives $s_1 : \boldsymbol{\beta} \longrightarrow A_1$ and $s_2 : \varphi_k(\boldsymbol{\beta}) \longrightarrow A_2$. There exists a unique isomorphism of Cohen rings*

$$\varphi = (\varphi_A, \varphi_k) : (A_1, k_1) \longrightarrow (A_2, k_2),$$

which respects s_1 and s_2, and which is φ_k on the residue fields.

Proof. — If both (A_1, k_1) and (A_2, k_2) are strict then both existence and uniqueness follow from Theorem 6.2. Suppose next that both (A_1, k_1) and (A_2, k_2) are of characteristic p^m. Let (B, k_1) be a strict Cohen ring with representatives $s : \boldsymbol{\beta} \longrightarrow B$. By Theorem 6.2 there are unique morphisms $\varphi^1 = (\varphi_A^1, \mathrm{id}_{k_1}) : (B, k_1) \longrightarrow (A_1, k_1)$ and $\varphi^2 = (\varphi_A^2, \varphi_k) : (B, k_1) \longrightarrow (A_2, k_2)$ which respect s and s_1 (resp. s_2) and which induce the identity (resp. φ_k) on the residue field. Moreover, both φ_A^i are surjective and both factor through the quotient map $B \longrightarrow B/\mathfrak{m}^m$ (cf. Remark 6.3). Thus, by the Isomorphism Theorem, both (A_i, k_i) are isomorphic to $(B/\mathfrak{m}^m, k_1)$. Therefore there is an isomorphism between them that respects s_1 and s_2, and induces φ_k on the residue field. This isomorphism is unique, by Theorem 6.2. $\qquad\square$

As long as one is only interested in the existence of an isomorphism of Cohen rings, the following simplified version of the above is sufficient:

COROLLARY 6.6 (Cohen Structure Theorem, v.2). — *Let (A_1, k_1) and (A_2, k_2) be Cohen rings of the same characteristic, and assume that $\varphi_k : k_1 \longrightarrow k_2$ is an isomorphism of the residue fields. There exists an isomorphism of Cohen rings*

$$\varphi = (\varphi_A, \varphi_k) : (A_1, k_1) \longrightarrow (A_2, k_2),$$

which is φ_k on the residue fields.

Proof. — Immediate from Corollary 6.5. $\qquad\square$

Our aim is now to apply Cohen's Homomorphism Theorem to give a clear statement of the relative structure of Cohen rings. That is, we will describe the morphisms between Cohen rings which extend a given morphism between subrings. Although we will not refer to the statement later on in this paper, we state and

prove it for future reference. It should be noted once again that this is closely based on the work of Teichmüller, Mac Lane, Cohen, and others. See for example [23], [16], and [4].

THEOREM 6.7 (Relative Homomorphism Theorem). — *Let $(A_1, k_1) \subseteq (A_2, k_2)$ and $(B_1, l_1) \subseteq (B_2, l_2)$ be two extensions of Cohen rings, let*

$$\varphi = (\varphi_A, \varphi_k) : (A_1, k_1) \longrightarrow (B_1, l_1)$$

be a morphism, and let $\Phi_k : k_2 \longrightarrow l_2$ be an embedding of fields which extends φ_k. Suppose that both $l_2/\Phi_k(k_2)$ and k_2/k_1 are separable. Let $\boldsymbol{\beta}$ be a p-basis of k_2 over k_1, and let $s_A : \boldsymbol{\beta} \longrightarrow A_2$ and $s_B : \Phi_k(\boldsymbol{\beta}) \longrightarrow B_2$ be choices of representatives.

Then, there exists a unique morphism of Cohen rings

$$\Phi := (\Phi_A, \Phi_k) : (A_2, k_2) \longrightarrow (B_2, l_2),$$

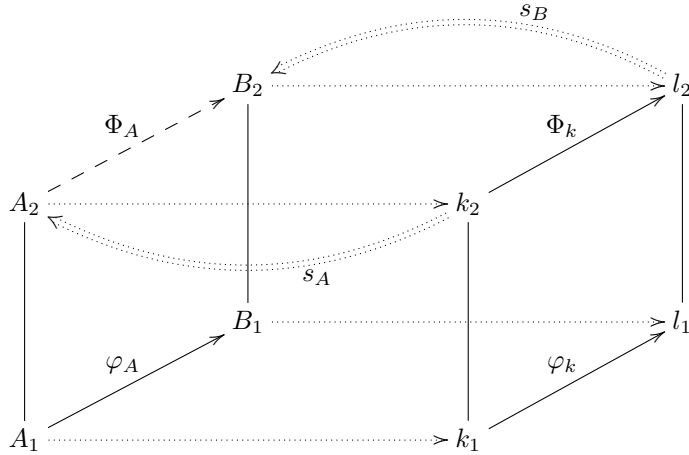*that respects s_A and s_B, that induces Φ_k on the residue fields, and that extends φ.*

FIGURE 6.2. Illustration of Theorem 6.7

Proof. — We are given a p-basis $\boldsymbol{\beta}$ of k_2 over k_1, that is each $\beta \in \boldsymbol{\beta}$ is of degree p over $k_2^{(p)} k_1(\boldsymbol{\beta} \setminus \{\beta\})$, and $k_2 = k_2^{(p)} k_1(\boldsymbol{\beta})$. Choose any p-basis $\boldsymbol{\beta}_{A,1}$ of k_1 and any representatives $s_{A,1} : \boldsymbol{\beta}_{A,1} \longrightarrow A_1$. Since k_2/k_1 is separable, $\boldsymbol{\beta}_{A,2} := \boldsymbol{\beta} \sqcup \boldsymbol{\beta}_{A,1}$ is a p-basis of k_2. We define

$$s_{A,2} : \boldsymbol{\beta}_{A,2} \longrightarrow A_2$$
$$\beta \longmapsto \begin{cases} s_{A,1}(\beta) & \beta \in \boldsymbol{\beta}_{A,1} \\ s_A(\beta) & \beta \in \boldsymbol{\beta}, \end{cases}$$

which is a choice of representatives for $\boldsymbol{\beta}_{A,2}$. Next we let $\boldsymbol{\beta}_{B,2} := \varphi_k(\boldsymbol{\beta}_{A,1}) \sqcup \Phi_k(\boldsymbol{\beta})$. We define

$$s_{B,2} : \boldsymbol{\beta}_{B,2} \longrightarrow B_2$$
$$\beta \longmapsto \begin{cases} \varphi_A(s_{A,1}(\varphi_k^{-1}(\beta))) & \beta \in \varphi_k(\boldsymbol{\beta}_{A,1}) \\ s_B(\beta) & \beta \in \Phi_k(\boldsymbol{\beta}), \end{cases}$$

which is a choice of representatives for $\boldsymbol{\beta}_{B,2}$. It follows from Theorem 6.2 that there is a unique morphism

$$\Phi = (\Phi_A, \Phi_k) : (A_2, k_2) \longrightarrow (B_2, k_2),$$

which respects $s_{A,2}$ and $s_{B,2}$, and which is Φ_k on the residue fields. Observe that Φ extends φ since in particular Φ respects $s_{A,1}$ and $\varphi_A \circ s_{A,1} \circ \varphi_k^{-1}|_{\varphi_k(\boldsymbol{\beta}_{A,1})}$ (the latter being a choice of representatives for $\varphi_k(\boldsymbol{\beta}_{A,1})$). This proves the existence part of our claim. For uniqueness, if Ψ is any other morphism which extends φ and respects s_A and s_B then we may argue that it also respects $s_{A,2}$ and $s_{B,2}$, just as for Φ. Therefore $\Phi = \Psi$ by Theorem 6.2. $\hspace{1cm}\square$

7. Cohen-Witt rings

Let k denote a field of characteristic $p > 0$. For each natural number $n \in \mathbb{N}$, we denote the n-**th Witt ring** over k by $W_{n+1}(k)$, and the **infinite Witt ring** we denote by $W[k]$, as described, for example, in [25] and in many other places.

If k is perfect, then $W[k]$ is a complete discrete valuation ring of characteristic zero with residue field k. That is, $(W[k], k)$ is a strict Cohen ring. By Theorem 6.2, $(W[k], k)$ may be viewed as providing the canonical example of a Cohen ring with residue field k, canonical in the sense that for perfect k there is a canonical isomorphism between any two strict Cohen rings with residue field k. Likewise, $(W_n(k), k)$ is the canonical example of a Cohen ring with residue field k, of characteristic p^n.

On the other hand, if k is imperfect, then $W[k]$ fails to be a valuation ring. There is a less well-known construction, appropriate for the case of imperfect residue fields, which constructs Cohen rings as subrings of Witt rings (see e.g. [19]). To mitigate the conflict with our own terminology, we will refer to these more concrete rings as 'Cohen-Witt rings'. We fix a p-basis $\boldsymbol{\beta}$ of k. For each $n \in \mathbb{N}$, the n-**th Cohen-Witt ring** over k, which we denote by $C_{n+1}(k)$, is the subring of $W_{n+1}(k)$ generated by $W_{n+1}(k^{(p^n)})$ and the elements $[\beta] = (\beta, 0, ...)$, for $\beta \in \boldsymbol{\beta}$. That is

$$C_{n+1}(k) := W_{n+1}(k^{(p^n)})\big([\beta] \mid \beta \in \boldsymbol{\beta}\big).$$

We note that $C_{n+1}(k)$ is a local ring, with maximal ideal (p) and residue field k. Thus $(C_{n+1}(k), k)$ is indeed a Cohen ring. There are representatives $s_n : \boldsymbol{\beta} \longrightarrow C_{n+1}(k)$, given by $s_n(\beta) = [\beta]$, for $\beta \in \boldsymbol{\beta}$. The maps $\pi_n : W_{n+1}(k) \longrightarrow W_n(k)$, which are given by the truncation of the Witt vectors, restrict to surjections

$$\pi_n|_{C_{n+1}(k)} : C_{n+1}(k) \longrightarrow C_n(k).$$

Just as with the Witt rings, the Cohen-Witt rings equipped with these maps form an inverse system, as in Remark 2.11, the inverse limit of which is the **strict Cohen-Witt ring** over k:

$$C[k] := \varprojlim C_{n+1}(k).$$

The field of fractions of $C[k]$ is often denoted by $C(k)$. This system may be enriched with a compatible system of representatives $s_n : \boldsymbol{\beta} \longrightarrow C_{n+1}(k)$, as in Remark 3.7. It is a consequence of Corollary 6.6 that any strict Cohen ring (A, k) is isomorphic to the strict Cohen-Witt ring $C[k]$, though the isomorphism is not canonical in the sense that it depends on our choices of $\boldsymbol{\beta}$ and s.

Part 2. **The Model Theory**

8. Theories and Completeness

Having developed the algebraic theory of Cohen rings, we are now in a position to describe their first-order theories. Let $\mathfrak{L}_{\mathrm{ring}} = \{+, -, \cdot, 0, 1\}$ denote the first-order language of rings, and $\mathfrak{L}_{\mathrm{vf}} = \mathfrak{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ the expansion of $\mathfrak{L}_{\mathrm{ring}}$ by a unary predicate (usually interpreted as the valuation ring). To study the structure of the value group, we also consider the language of ordered abelian groups $\mathfrak{L}_{\mathrm{oag}} = \{0, +, \leqslant\}$.

We consider the following theories:

DEFINITION 8.1. — Let k be a field of characteristic p.
- Let T_{pc} be the $\mathfrak{L}_{\mathrm{ring}}$-theory of commutative rings with unity in which (p) is the unique maximal ideal.
- Let $T_{\mathrm{pc}}(k, n)$ be the $\mathfrak{L}_{\mathrm{ring}}$-theory consisting of the union of T_{pc} with axioms that assert of a model B that its characteristic is p^n and that its residue field k_B is a elementarily equivalent to k.
- Let T_{ur} be the $\mathfrak{L}_{\mathrm{vf}}$-theory that asserts of a model (K, v) that v is henselian, $\mathcal{O}_v$ is a valuation ring on K which is a model of T_{pc}, and that the characteristic of K is zero.
- Let $T_{\mathrm{ur}}(k, \Gamma)$ be the $\mathfrak{L}_{\mathrm{vf}}$-theory extending T_{ur} which requires in addition for a model (K, v) that Kv is elementarily equivalent to k and that vK is elementarily equivalent to Γ.

Note that if Γ is an ordered abelian group *without minimum positive element* then $T_{\mathrm{ur}}(k, \Gamma)$ is not consistent. We do not define $T_{\mathrm{ur}}(k, \Gamma)$ for a field k of characteristic 0.

The aim of this section is to show that $T_{\mathrm{pc}}(k, n)$ and $T_{\mathrm{ur}}(k, \Gamma)$ are complete and to deduce the usual AKE type consequences from this. The non-strict case is a simple application of the Structure Theorem for Cohen rings of positive characteristic:

THEOREM 8.2. — *For any field k of positive characteristic, the $\mathfrak{L}_{\mathrm{ring}}$-theory $T_{\mathrm{pc}}(k, n)$ is complete.*

Proof. — Let $B_1, B_2 \models T_{\mathrm{pc}}(k, n)$. By the Keisler-Shelah Theorem ([21]), replacing B_1 and B_2 with suitable ultrapowers if necessary, we may assume that there is an $\mathfrak{L}_{\mathrm{ring}}$-isomorphism $\varphi_k : k_{B_1} \longrightarrow k_{B_2}$. Applying the Structure Theorem for Cohen rings (Corollary 6.6), we get an isomorphism $\varphi : B_1 \longrightarrow B_2$ that induces φ_k. In particular this implies that B_1 and B_2 are elementarily equivalent. $\square$

For the general case, we use the usual proof method of combining the Structure Theorem with a coarsening argument which allows us to apply the Ax-Kochen/Ershov Theorem in the equicharacteristic 0 case:

THEOREM 8.3. — *For any field k of positive characteristic and any ordered abelian group Γ with minimum positive element, the $\mathfrak{L}_{\mathrm{vf}}$-theory $T_{\mathrm{ur}}(k, \Gamma)$ is complete.*

Proof. — Let $(K_1, v_1), (K_2, v_2) \models T_{\mathrm{ur}}(k, \Gamma)$. By the Keisler-Shelah Theorem ([21]), replacing each structure with a suitable ultrapower if necessary, we may assume that both (K_1, v_1) and (K_2, v_2) are $\aleph_1$-saturated, and that there is an $\mathfrak{L}_{\mathrm{ring}}$-isomorphism $\varphi_k : K_1 v_1 \longrightarrow K_2 v_2$ and an $\mathfrak{L}_{\mathrm{oag}}$-isomorphism $\varphi_\Gamma : v_1 K_1 \longrightarrow v_2 K_2$.

For $i = 1, 2$, let w_i denote the finest proper coarsening of v_i (note that w_i exists because $v_i K_i$ has a minimum positive element) and let $\bar{v}_i$ denote the valuation induced by v_i on the residue field $K_i w_i$.

By $\aleph_1$-saturation, both valued fields $(K_i w_i, \bar{v}_i)$ are spherically complete and hence the valuation rings $\mathcal{O}_{\bar{v}_i}$ are strict Cohen rings. By the Structure Theorem for Cohen rings (Corollary 6.6), there exists an isomorphism $\varphi : \mathcal{O}_{\bar{v}_1} \longrightarrow \mathcal{O}_{\bar{v}_2}$ that induces φ_k. Note that φ_Γ also induces an isomorphism $\bar{\varphi}_\Gamma : w_1 K_1 \longrightarrow w_2 K_2$ because any isomorphism of ordered abelian groups sends a minimum positive element (and hence the generated convex subgroup) to another such. Since both (K_i, w_i) are henselian of equicharacteristic zero, the Ax-Kochen/Ershov principle ([25, AKE-Theorem 5.1]) implies that (K_1, w_1) and (K_2, w_2) are elementarily equivalent.

It follows by the $\emptyset$-definability of the valuation v_i in K_i (this is a variant of Robinson's classical definition of $\mathbb{Z}_p$ in $\mathbb{Q}_p$, see, e.g., [6, Corollary 2]) that (K_1, v_1) and (K_2, v_2) are elementarily equivalent, as required. $\square$

Remark 8.4. — In fact, Bélair proves relative quantifier elimination for the valued field sort of an unramified henselian valued field in the ω-sorted language $\mathcal{L}_{co_\omega}$ down to the sorts for the residue rings $\mathcal{O}/\mathfrak{m}^n$ (cf. [2, Théorème 5.1]). Applying this quantifier elimination, Theorem 8.3 can also be deduced from Theorem 8.2.

In particular, Theorem 8.3 immediately implies the following Ax-Kochen/Ershov-type result, sometimes referred to as an AKE$_\equiv$-principle:

COROLLARY 8.5 (Ax-Kochen/Ershov principle for unramified henselian valued fields). — *Let (K, v) and (L, w) be two unramified henselian valued fields. Then*

$$\underbrace{Kv \equiv Lw}_{in\ \mathfrak{L}_{ring}} \ and \ \underbrace{vK \equiv wL}_{in\ \mathfrak{L}_{oag}} \Longleftrightarrow \underbrace{(K, v) \equiv (L, w)}_{in\ \mathfrak{L}_{vf}}.$$

Remark 8.6. — Corollary 8.5 above is essentially claimed by Bélair in [2, Corollaire 5.2]. However, Bélair's proof only goes through in the case of a perfect residue field, since it uses the rings of Witt vectors.

The Ax-Kochen/Ershov Principle, above, immediately gives an axiomatisation of the complete theories of unramified henselian valued fields, as follows.

COROLLARY 8.7. — *Let (K, v) be an unramified henselian valued field of mixed characteristic. The complete $\mathfrak{L}_{vf}$-theory of (K, v) is axiomatised by*

  (i) *(K, v) is an henselian valued field of mixed characteristic $(0, p)$,*

  (ii) *the value group is elementarily equivalent to vK and $v(p)$ is minimum positive, and*

  (iii) *the residue field is elementarily equivalent to Kv.*

In particular, we get an axiomatization of the $\mathfrak{L}_{vf}$-theory of $(C(k), v)$, relative to the $\mathfrak{L}_{ring}$-theory of the residue field k.

9. RELATIVE MODEL COMPLETENESS

In analogy to the case of unramified henselian fields with perfect residue field (cf. [25, Theorem 7.2]), we also get an AKE$_\preceq$-principle for the case of arbitrary residue fields. To prove relative completeness, or in other words an AKE$_\equiv$-principle

for unramified henselian valued fields, it is sufficient to know that Cohen rings are unique up to isomorphism. However, in order to prove the AKE$_\preceq$-principle for unramified henselian valued fields, we need to apply the Relative Structure Theorem (Corollary 6.4). We first state the non-strict version:

PROPOSITION 9.1 (Relative model completeness, non-strict version). — *Given* $A, B \models T_{\mathrm{pc}}(k, n)$ *with* $A \subseteq B$ *such that the induced embedding of residue fields* $k_A \subseteq k_B$ *is an elementary embedding (in* $\mathfrak{L}_{\mathrm{ring}}$), *we have*

$$A \preceq B.$$

Proof. — Let $B_1, B_2, A \models T_{\mathrm{pc}}(k, n)$ with $A \subseteq B_1, B_2$ two extensions. We suppose that the induced extensions $k_A \preceq k_{B_i}$, for $i = 1, 2$, are elementary. We claim that B_1 and B_2 are elementarily equivalent over A, symbolically $B_1 \equiv_A B_2$.

By the Keisler-Shelah Theorem ([21]), replacing B_1 and B_2 with suitable ultrapowers if necessary, we may assume that there is an $\mathfrak{L}_{\mathrm{ring}}$-isomorphism $\varphi_k : k_{B_1} \longrightarrow k_{B_2}$ that fixes k_A pointwise. By Corollary 6.4, there is an isomorphism $\varphi : (B_1, k_1) \longrightarrow (B_2, k_2)$ that induces φ_k and fixes A pointwise. In particular, B_1 and B_2 are elementarily equivalent over A. This proves the claim.

We return to the setting of an extension $A \subseteq B$ of models of $T_{\mathrm{pc}}(k, n)$ for which $k_A \preceq k_B$ is elementary. From the claim it follows that $A \equiv_A B$, equivalently the extension $A \subseteq B$ is elementary. $\qquad\square$

For the strict version of the relative model completeness theorem, we combine the Relative Structure Theorem (Corollary 6.4) with the coarsening method and well-known results from the equicharacteristic zero world.

THEOREM 9.2 (Relative model completeness). — *Given an extension* $(K, v) \subseteq (L, w)$ *of unramified henselian valued fields such that the induced embeddings of residue fields* $Kv \subseteq Lw$ *and value groups* $vK \subseteq wL$ *are elementary (in* $\mathfrak{L}_{\mathrm{ring}}$ *and* $\mathfrak{L}_{\mathrm{oag}}$ *respectively), we have*

$$(K, v) \preceq (L, w).$$

Proof. — Let $(K_i, v_i) \models T_{\mathrm{ur}}(k, \Gamma)$, for $i = 0, 1, 2$, be such that $(K_0, v_0) \subseteq (K_1, v_1), (K_2, v_2)$ are two extensions of valued fields. We suppose that the residue field extensions $K_0 v_0 \preceq K_i v_i$, and the value group extensions $v_0 K_0 \preceq v_i K_i$, both for $i = 1, 2$, are elementary. We claim that (K_1, v_1) and (K_2, v_2) are elementarily equivalent over (K_0, v_0), symbolically $(K_1, v_1) \equiv_{(K_0, v_0)} (K_2, v_2)$.

By the Keisler-Shelah Theorem ([21]), replacing each valued field with a suitable ultrapower if necessary, we may assume that all three valued fields are $\aleph_1$-saturated and that there is an isomorphism $\varphi_k : K_1 v_1 \longrightarrow K_2 v_2$ that fixes $K_0 v_0$ pointwise, and an isomorphism $\varphi_\Gamma : v_1 K_1 \longrightarrow v_2 K_2$ that fixes $v_0 K_0$ pointwise. For $i = 0, 1, 2$, let $\hat{v}_i$ be the finest proper coarsening of v_i, and let $\bar{v}_i$ be the valuation induced on $K_i \hat{v}_i$ by v_i. By $\aleph_1$-saturation, all three $(K_i \hat{v}_i, \bar{v}_i)$ are strict Cohen rings. Note also that these coarsenings are compatible in the sense that, for both $i = 1, 2$, we have extensions $(K_0, \hat{v}_0) \subseteq (K_i, \hat{v}_i)$ and $(K_0 \hat{v}_0, \bar{v}_0) \subseteq (K_i \hat{v}_i, \bar{v}_i)$. Moreover φ_Γ induces an isomorphism $\hat{\varphi}_\Gamma : \hat{v}_1 K_1 \longrightarrow \hat{v}_2 K_2$ that fixes $\hat{v}_0 K_0$ pointwise, since φ_Γ restricts to an isomorphism between the convex subgroups $\langle v_1(p) \rangle$ and $\langle v_2(p) \rangle$, and since $\hat{v}_i K_i$ is the quotient of $v_i K_i$ by $\langle v_i(p) \rangle$.

By Corollary 6.4, there is an isomorphism $\varphi : (K_1\hat{v}_1, \bar{v}_1) \longrightarrow (K_2\hat{v}_2, \bar{v}_2)$ that induces φ_k and fixes $K_0\hat{v}_0$ pointwise. In particular $K_1\hat{v}_1$ and $K_2\hat{v}_2$ are elementarily equivalent over $K_0\hat{v}_0$.

Therefore $(K_1, \hat{v}_1)$ and $(K_2, \hat{v}_2)$ are two henselian valued fields of equicharacteristic zero, both extending $(K_0, \hat{v}_0)$, with value groups isomorphic over $\hat{v}_0 K_0$ and residue fields elementarily equivalent over $K_0\hat{v}_0$. It follows from [12, Theorem 7.1] that $(K_1, \hat{v}_1)$ and $(K_2, \hat{v}_2)$ are elementarily equivalent over $(K_0, \hat{v}_0)$, i.e. $(K_1, \hat{v}_1) \equiv_{(K_0,\hat{v}_0)} (K_2, \hat{v}_2)$. Since the valuation rings of all three valuations v_i are $\emptyset$-definable by the same $\mathfrak{L}_{\mathrm{ring}}$-formula in each field K_i (again, this is a variant of Robinson's classical definition of $\mathbb{Z}_p$ in $\mathbb{Q}_p$, see, e.g., [6, Corollary 2]), it follows that (K_1, v_1) and (K_2, v_2) are elementarily equivalent over (K_0, v_0), i.e. $(K_1, v_1) \equiv_{(K_0,v_0)} (K_2, v_2)$. This proves the claim.

We return to the setting of an extension $(K, v) \subseteq (L, w)$ of models of T_{ur} for which $Kv \preceq Lw$ and $vK \preceq wL$ are elementary. From the claim it follows that $(K, v) \equiv_{(K,v)} (L, w)$, equivalently the extension $(K, v) \subseteq (L, w)$ is elementary. $\square$

Remark 9.3. — In fact, the relative model completeness for non-strict Cohen rings (Proposition 9.1) can also be combined with a result by Bélair ([2, Corollaire 5.2(2)]) to show model completeness, albeit in a slightly different (ω-sorted) language.

As a consequence, we get the following embedding version of the Ax-Kochen/ Ershov result:

COROLLARY 9.4. — *Let $(K, v) \subseteq (L, w)$ be two unramified henselian valued fields. Then, we have*

$$\underbrace{Kv \preceq Lw}_{in\ \mathfrak{L}_{\mathrm{ring}}} \text{ and } \underbrace{vK \preceq wL}_{in\ \mathfrak{L}_{\mathrm{oag}}} \Longleftrightarrow \underbrace{(K, v) \preceq (L, w)}_{in\ \mathfrak{L}_{\mathrm{vf}}}.$$

For Cohen rings, or more generally unramified henselian valued fields taking values in a $\mathbb{Z}$-group, our result simplifies to:

COROLLARY 9.5. — *Let $(K, v) \subseteq (L, w)$ be two unramified henselian valued fields with value groups $vK \equiv wL \equiv \mathbb{Z}$. Then, we have*

$$\underbrace{Kv \preceq Lw}_{in\ \mathfrak{L}_{\mathrm{ring}}} \Longleftrightarrow \underbrace{(K, v) \preceq (L, w)}_{in\ \mathfrak{L}_{\mathrm{vf}}}.$$

10. Embedding lemma and relative existential completeness

The aim of this section is to prove an embedding lemma for unramified henselian valued fields. This will be applied to prove relative existential completeness of unramified henselian valued fields of fixed finite degree of imperfection as well as to show stable embeddedness of residue field and value group in the subsequent section. The proof of the embedding lemma (i.e. the next proposition) is a refined version of the proofs given in [12, Lemmas 5.6 and 6.4]. In Kuhlmann's terminology, we show that the class of $\aleph_1$-saturated models of T_{ur} satisfies an appropriate version of the Relative Embedding Property (cf. [12, p. 31]).

PROPOSITION 10.1 (Embedding Lemma). — *Let (L_1, v_1) and (L_2, v_2) be extensions of (K, v), and assume that all three are $\aleph_1$-saturated models of T_{ur}. Suppose that $L_1 v_1 / Kv$ is regular and $v_1 L_1 / vK$ is torsion-free. Moreover, assume that*

(L_2, v_2) is $|L_1|^+$-saturated and that there are embeddings $\varphi_k : L_1 v_1 \longrightarrow L_2 v_2$ over Kv and $\varphi_\Gamma : v_1 L_1 \longrightarrow v_2 L_2$ over vK. Suppose that $L_2 v_2 / \varphi_k(L_1 v_1)$ is separable. Then, there is an embedding $\varphi : (L_1, v_1) \longrightarrow (L_2, v_2)$ over K that induces φ_k and φ_Γ on residue field and value group respectively.

Moreover, if φ_k and φ_Γ are elementary embeddings, then any such embedding φ is elementary.

Proof. — Since (K, v) is henselian and unramified, it is defectless, therefore our assumptions on value group and residue field imply that K is relatively algebraically closed in L_1. Let w denote the finest proper coarsening of v on K, and, likewise, let w_i denote the finest proper coarsening of v_i on L_i (for $i = 1, 2$). Note that w is the restriction of each w_i to K. By our saturation assumption, the valued fields $(Kw, \bar{v})$ and $(L_i w_i, \bar{v}_i)$ are all Cohen fields. The inclusion $(K, w) \subseteq (L_1, w_1)$ gives rise to an inclusion $Kw \subseteq L_1 w_1$. The embedding φ_Γ induces an embedding $\psi_\Gamma : w_1 L_1 \longrightarrow w_2 L_2$ over wK. Moreover, φ_k induces an (not necessarily unique) embedding

$$\psi_k : (L_1 w_1, \bar{v}_1) \longrightarrow (L_2 w_2, \bar{v}_2)$$

over $(Kw, \bar{v})$ by Cohen's Homomorphism theorem (Theorem 6.2). We now fix one such ψ_k. We note that, in order to show that an embedding φ of a subfield of (L_1, v_1) into (L_2, v_2) induces φ_k, it suffices to show that it induces ψ_k:

CLAIM 1. — *For any $(F, w_1, v_1) \subseteq (L_1, w_1, v_1)$ extending (K, w, v), if*

$$\varphi : (F, w_1) \longrightarrow (L_2, w_2)$$

is an embedding over K that induces ψ_k, then φ is also an embedding

$$\varphi : (F, v_1) \longrightarrow (L_2, v_2)$$

that induces φ_k.

Proof of claim. — Note that ψ_k is an embedding of valued fields $(L_1 w_1, \bar{v}_1) \longrightarrow (L_2 w_2, \bar{v}_2)$. For $a \in F$, we have

$$
\begin{aligned}
v_1(a) \geqslant 0 &\Leftrightarrow w_1(a) \geqslant 0 \text{ and } \bar{v}_1(aw_1) \geqslant 0 \\
&\Leftrightarrow w_2(\varphi(a)) \geqslant 0 \text{ and } \bar{v}_2(\psi_k(aw_1)) \geqslant 0 \\
&\Leftrightarrow w_2(\varphi(a)) \geqslant 0 \text{ and } \bar{v}_2((\varphi(a))w_2) \geqslant 0 \\
&\Leftrightarrow v_2(\varphi(a)) \geqslant 0.
\end{aligned}
$$

Since φ induces ψ_k, and ψ_k is an embedding $(Fw_1, \bar{v}_1) \longrightarrow (L_2 w_2, \bar{v}_2)$ that induces φ_k, it follows that φ induces φ_k. $\qquad\square_{\text{claim}}$

We now adapt the proof of [12, Lemmas 5.6 and 6.4] carefully to our setting, in order to construct an embedding $\varphi : (L_1, v_1) \longrightarrow (L_2, v_2)$ over K that induces φ_k and φ_Γ. We also use φ to denote the restriction of φ to any subfield of L_1. Let $\mathcal{T}$ be a standard valuation transcendence basis of $(L_1, w_1)/(K, w)$, i.e.

$$\mathcal{T} = \{x_i, y_i \mid i \in I, j \in J\}$$

such that the set of values $\{w_1(x_i)\}_{i \in I}$ is a maximal rationally independent set in $w_1 L$ over wK and such that the set of residues $\{y_j w_1\}_{j \in J}$ is a transcendence base of $L_1 w_1$ over Kw. Let K' be the relative algebraic closure of $K(\mathcal{T})$ in L_1, and, by an abuse of notation, let w_1 also denote the restriction of w_1 to K' and its subfields.

Note that $(K', w_1)/(K, w)$ is without transcendence defect, by [12, Corollary 2.4]. See [12, p. 4] for the definition of 'without transcendence defect'.

Step 1: Extending to valued function fields without transcendence defect. Let L be a subfield of K' that is a finitely generated extension of K. Since K is relatively algebraically closed in L, $(L, w_1)/(K, w)$ is a valued function field without transcendence defect. By [12, Theorem 1.9], $(L, w_1)/(K, w)$ is strongly inertially generated, i.e. there is a transcendence basis $\mathcal{T}_L = \{x_i, y_j \mid i \in I_L, j \in J_L\} \subseteq L$ such that

(i) $w_1 L = w_1 K(\mathcal{T}_L) = wK \oplus \bigoplus_i \mathbb{Z} \cdot w_1(x_i)$

(ii) $\{y_j w_1\}_{j \in J_L}$ is a separating transcendence base of Lw_1/Kw, and

(iii) there is an element $a \in L^h$ (the henselization of L with respect to w_1) such that $L^h = K(\mathcal{T}_L)^h(a)$, $w_1(a) = 0$, and $(K(\mathcal{T}_L)w_1)(aw_1)/K(\mathcal{T}_L)w_1$ is a separable extension of degree $[K(\mathcal{T}_L)^h(a) : K(\mathcal{T}_L)^h]$.

Note that in our case, the separability in (ii) and (iii) is automatic, since w_1 is of residue characteristic zero. We now explore these three properties, one after the other, in order to construct an embedding $\varphi : (L, v_1) \longrightarrow (L_2, v_2)$ over K that induces φ_k and φ_Γ.

CLAIM 2. — $v_1 L = vK \oplus \bigoplus_i \mathbb{Z} \cdot v_1(x_i)$.

Proof of claim. — Suppose there are $n_i \in \mathbb{Q}$ and $\alpha \in vK$ such that $\alpha + \sum_i n_i \cdot v_1(x_i) = 0$. Then $(\alpha + \mathbb{Z}) + \sum_i n_i \cdot w_1(x_i) = 0 + \mathbb{Z}$. By $\mathbb{Q}$-linear independence of the $w_1(x_i)$ over wK in $w_1 L$, all the n_i are zero. Therefore the $v_1(x_i)$ are $\mathbb{Q}$-linearly independent from vK in $v_1 L$. Thus $vK \oplus \bigoplus_i \mathbb{Z} \cdot v_1(x_i) \leqslant v_1 L$.

Let $\gamma \in v_1 L$. There exist $n_i \in \mathbb{Z}$ and $\alpha \in vK$ such that $(\gamma + \mathbb{Z}) = (\alpha + \mathbb{Z}) + \sum_i n_i w_1(x_i)$. Therefore for some $\beta \in vK$ we have $\gamma = \beta + \sum_i n_i v_1(x_i)$. $\square_{\text{claim}}$

Next, we choose $\mathcal{T}_L' = \{x_i', y_j' \mid i \in I_L, j \in J_L\} \subseteq L_2$ such that

(i) $v_2(x_i') = \varphi_\Gamma(v_1(x_i))$, for each $i \in I_L$, and

(ii) $y_j' w_2 = \psi_k(y_j w_1)$, for each $j \in J_L$.

Immediately: $w_2(x_i') = \psi_\Gamma(w_1(x_i))$, for each $i \in I_L$. Exactly as in the proof of [12, Lemma 5.6], there is an isomorphism $\varphi : (K(\mathcal{T}_L), w_1) \longrightarrow (K(\mathcal{T}_L'), w_2)$ that maps

$$x_i \longmapsto x_i'$$
$$y_j \longmapsto y_j',$$

and which therefore induces ψ_k and ψ_Γ.

CLAIM 3. — φ is also an isomorphism $\varphi : (K(\mathcal{T}_L), v_1) \longrightarrow (K(\mathcal{T}_L'), v_2) \subseteq (L_2, v_2)$ which induces φ_k and φ_Γ.

Let $f \in K[\mathcal{T}_L]$, written $f = \sum_k c_k \prod_i x_i^{\mu_{k,i}} \prod_j y_j^{\nu_{k,j}}$, for $c_k \in K$.

Proof of claim. — We already know that φ is an isomorphism of fields, inducing both ψ_K and ψ_Γ. By Claim 1, φ is moreover an isomorphism of valued fields with respect to the v_i's that induces φ_k.

We now check that it induces φ_Γ, first working in the case $f \in K[x_i \mid i \in I_L]$ and using the $\mathbb{Q}$-linear independence over vK of the $v_1 x_i$, and of the $v_2 x_i'$:

$$v_2(\varphi(f)) = \min_k\{v_2(c_k) + \sum_i \mu_{k,i} v_2 x_i'\}$$

$$= \min_k\{v_2(c_k) + \sum_i \mu_{k,i} \varphi_\Gamma(v_1 x_i)\}$$

$$= \varphi_\Gamma(\min_k\{v_1(c_k) + \sum_i \mu_{k,i} v_1 x_i\})$$

$$= \varphi_\Gamma(v_1(f)).$$

Since the value group of $K[x_i \mid i \in I_L]$ with respect to v_1 is already $vK \oplus \bigoplus_i \mathbb{Z} \cdot v_1 x_i = v_1 K(\mathcal{T}_L)$, indeed φ induces φ_Γ. $\qquad\square_{\text{claim}}$

By the universal property of henselizations, φ extends to an isomorphism

$$\varphi : (K(\mathcal{T}_L)^h, w_1) \longrightarrow (K(\mathcal{T}_L')^h, w_2) \subseteq (L_2, w_2)$$

where the henselizations are taken with respect to w_i (for $i \in \{1, 2\}$). Note that φ still induces both ψ_k and ψ_Γ since henselizations are immediate extensions. Thus, by Claim 1, φ is also an isomorphism $(K(\mathcal{T}_L)^h, v_1) \longrightarrow (K(\mathcal{T}_L')^h, v_2)$. Since the henselization $K(\mathcal{T}_L)^h$ of $K(\mathcal{T}_L)$ with respect to w_1 is a subfield of the henselization with respect to v_1, which is an immediate extension of $(K(\mathcal{T}_L), v_1)$, φ induces φ_k and φ_Γ.

Recall that by property **(iii)** of strong inertial generation, there exists an element $a \in L^h$ such that $L^h = K(\mathcal{T}_L)^h(a)$, $w_1(a) = 0$, and $(K(\mathcal{T}_L)^h w_1)(a w_1)/K(\mathcal{T}_L)^h w_1$ is a separable extension of degree $[K(\mathcal{T}_L)^h(a) : K(\mathcal{T}_L)^h]$.

Let $f \in \mathcal{O}_{(K(\mathcal{T}_L)^h, w_1)}[X]$ be such that fw_1 is the minimal polynomial of $a w_1$ over $K(\mathcal{T}_L)^h w_1$. By henselianity, there exists a unique root $a' \in L_2$ of $\varphi(f)$ such that $a' w_2 = \psi_k(a w_1)$.

CLAIM 4. — φ extends to an isomorphism

$$\varphi : (K(\mathcal{T}_L)^h(a), v_1) \longrightarrow (K(\mathcal{T}_L')^h(a'), v_2) \subseteq (L_2, v_2)$$

which maps a to a' and induces φ_k and φ_Γ.

Proof of claim. — Mapping $a \longmapsto a'$ allows us to extend φ to an isomorphism of fields $\varphi : K(\mathcal{T}_L)^h(a) \longrightarrow K(\mathcal{T}_L')^h(a')$. By henselianity of $(K(\mathcal{T}_L)^h, w_1)$, φ is an isomorphism of valued fields $(K(\mathcal{T}_L)^h(a), w_1) \longrightarrow (K(\mathcal{T}_L')^h(a'), w_2)$.

We now argue that φ induces ψ_k. Since $1, a w_1, \ldots, a^{n-1} w_1$ are linearly independent over $K(\mathcal{T}_L)^h w_1$, we have $w_1\left(\sum_{i<n} c_i a^i\right) = \min_i w_1(c_i)$, for all $c_i \in K(\mathcal{T}_L)^h$. In particular, we have

$$\mathcal{O}_{(K(\mathcal{T}_L)^h(a), w_1)} = \Big\{ \sum_{i<n} c_i a^i \,\Big|\, c_i \in \mathcal{O}_{(K(\mathcal{T}_L)^h, w_1)} \Big\}.$$

Let $g \in \mathcal{O}_{(K(\mathcal{T}_L)^h, w_1)}[X]$. Then φ induces ψ_k since we have

$$(\varphi(g(a))) w_2 = (\varphi(g)(a')) w_2 = (\varphi(g) w_2)(a' w_2)$$

$$= (\psi_k(g w_1))(\psi_k(a w_1)) = \psi_k((g(a)) w_1).$$

Thus, by Claim 1, $\varphi : (K(\mathcal{T}_L)^h(a), v_1) \longrightarrow (K(\mathcal{T}_L')^h(a'), v_2)$ is an isomorphism that induces φ_k. Finally, the value group of $(K(\mathcal{T}_L)^h(a), v_1)$ is torsion over the value

group of $(K(\mathcal{T}_L)^h, v_1)$: for each $\gamma \in v_1 K(\mathcal{T}_L)^h(a)$ we have $n\gamma \in v_1 K(\mathcal{T}_L)^h$. Since the restriction of φ to $K(\mathcal{T}_L)^h$ induces φ_Γ on $v_1 K(\mathcal{T}_L)^h$, and multiplication by n in ordered abelian groups is injective, we conclude that φ induces φ_Γ. $\square_{\mathrm{claim}}$

This finishes the construction of an embedding $\varphi : (L, v_1) \longrightarrow (L_2, v_2)$ that induces φ_k and φ_Γ. Therefore, we have completed Step 1.

By saturation, we obtain an embedding

$$\varphi : (K', v_1) \longrightarrow (L_2, v_2)$$

over K that induces φ_k and φ_Γ. More precisely, we realise the finitely consistent type

$$\mathrm{tp}_{\mathrm{qf}}((K', v_1)/(K, v)) \cup \{v(x_c) = \varphi_\Gamma(v_1(c)) \mid c \in K'\} \cup \{x_c v = \varphi_k(cv_1) \mid c \in \mathcal{O}_{(K', v_1)}\},$$

which is a type over $|K'|$-many parameters.

Step 2: Extending to immediate function fields. Our present aim is to extend φ to an embedding $(L_1, v_1) \longrightarrow (L_2, v_2)$ over K that induces φ_k and φ_Γ. Since K' contains a standard valuation transcendence basis for $(L_1, w_1)/(K, w)$, we have that $w_1 L_1 / w_1 K'$ is torsion and $L_1 w_1 / K' w_1$ is algebraic. Since K' is relatively algebraically closed in L_1, it follows that (K', v_1) and (K', w_1) are henselian. Because $\mathrm{char}(K' w_1) = 0$, we conclude that the extension $(L_1, w_1)/(K', w_1)$ is immediate (for example see [12, Lemma 3.7]). Thus, so is $(L_1, v_1)/(K', v_1)$. Therefore any extension of φ to an embedding $(L_1, v_1) \longrightarrow (L_2, v_2)$ automatically induces φ_k and φ_Γ.

Consider a finitely generated subextension F/K' of L_1/K'; then $(F, v_1)/(K', v_1)$ is an immediate function field. In fact, proceeding iteratively, it suffices to find a way to extend φ to immediate function fields of transcendence degree 1. By [13, Theorem 2.2], such extensions are henselian rational, i.e. subextensions of the henselization of a simple transcendental and immediate extension, of transcendental type.

Suppose we have extended φ to an embedding $\varphi : (F_0, v_1) \longrightarrow (L_2, v_1)$ over K that induces φ_k and φ_Γ, where (F_0, v_1) is the henselization of a finitely generated extension of K'. Let $b \in L_1$ be transcendental over F_0. Then b is a pseudo-limit of a pseudo-Cauchy sequence $(b_\rho)_{\rho < \sigma} \subseteq F_0$ of transcendental type, with respect to v_1: as (F_0, v_1) is henselian and algebraically maximal, all its immediate extensions are of transcendental type. Then $(\varphi(b_\rho))_{\rho < \sigma}$ is a pseudo-Cauchy sequence in L_2, also of transcendental type, now with respect to v_2. By saturation, this sequence has a pseudo-limit $b' \in L_2$. We extend φ by mapping $b \longmapsto b'$ to an isomorphism of fields $F_0(b) \longrightarrow \varphi(F_0)(b')$. Automatically, we get that

$$\varphi : (F_0(b), v_1) \longrightarrow (\varphi(F_0)(b'), v_2) \subseteq (L_2, v_2)$$

is an isomorphism of valued fields since pseudo-Cauchy sequences of transcendental type determine the isomorphism type of the valued field generated by a pseudo-limit (see [8, Theorem 2]).

Finally, we extend φ to the henselization of $(F_0(b), v_1)$, which is a subfield of (L_1, v_1). This is accomplished by applying the universal property of the henselization once again.

We have now constructed an embedding $\varphi : (L_1, v_1) \longrightarrow (L_2, v_2)$ over K which induces φ_k and φ_Γ. Last but not least, if both φ_k and φ_Γ are elementary embeddings, the AKE$_\preceq$-principle (Corollary 9.4) implies that φ is in fact an elementary embedding of unramified henselian valued fields, as desired. This finishes the proof. $\square$

THEOREM 10.2. — *Let $(K, v) \subseteq (L, w)$ be an extension of unramified henselian valued fields such that Kv and Lw have the same finite degree of imperfection. If the induced embeddings of residue field $Kv \subseteq Lw$ and value group $vK \subseteq wL$ are existentially closed (in $\mathfrak{L}_{ring}$ and $\mathfrak{L}_{oag}$ respectively), we have*

$$(K, v) \preceq_\exists (L, w).$$

Proof. — By taking ultrapowers if necessary, we may assume that both (K, v) and (L, w) are $\aleph_1$-saturated. Let (K^*, v^*) be a $|L|^+$-saturated elementary extension of (K, v). Since $Kv \preceq_\exists Lw$ and $vK \preceq_\exists wL$, there are embeddings $\varphi_k : Lw \longrightarrow (Kv)^* = K^*v^*$ over Kv and $\varphi_\Gamma : wL \longrightarrow (vK)^* = v^*K^*$ over vK. Let $\boldsymbol{\beta} \subseteq Kv$ be a p-basis. Since $Kv \preceq_\exists Lw$, $\boldsymbol{\beta}$ is also p-independent in Lw; in particular Lw/Kv is separable. Since the degree of imperfection of Lw is the same as that of Kv, $\boldsymbol{\beta}$ is a p-basis of Lw. Since K^*v^* is an elementary extension of Kv, $\boldsymbol{\beta}$ is also a p-basis of K^*v^*. Therefore $K^*v^*/\varphi_k(Lw)$ is separable. Finally, since $vK \preceq_\exists wL$, the group wL/vK is torsion-free. Thus we may apply Proposition 10.1 to obtain an embedding $\varphi : (L, w) \longrightarrow (K^*, v^*)$ over K that induces φ_k and φ_Γ. In particular, $(K, v) \preceq_\exists (L, w)$. $\square$

Note that the reason why we require a fixed finite degree of imperfection in Theorem 10.2 is that when we embed Lw into an elementary extension of Kv in the proof, we need to make sure that this latter extension is separable in order to apply Proposition 10.1.

As a consequence of Theorem 10.2, we get the following existential version of the Ax-Kochen/Ershov result:

COROLLARY 10.3. — *Let $(K, v) \subseteq (L, w)$ be two unramified henselian valued fields such that Kv and Lw have the same finite degree of imperfection. Then, we have*

$$\underbrace{Kv \preceq_\exists Lw}_{in \ \mathfrak{L}_{ring}} \ and \ \underbrace{vK \preceq_\exists wL}_{in \ \mathfrak{L}_{oag}} \iff \underbrace{(K, v) \preceq_\exists (L, w)}_{in \ \mathfrak{L}_{vf}}.$$

11. STABLE EMBEDDEDNESS

In this final section, we comment on the structure induced on the residue field and value group in unramified henselian valued fields.

DEFINITION 11.1. — *Let $\mathcal{M}$ be a structure and let $P \subseteq M^k$ be a definable set. We say that P is stably embedded if for all formulas $\varphi(x, y)$ and all $b \in M^{|y|}$, $\varphi(M^{k|x|}, b) \cap P^{|x|}$ is P-definable.*

In order to be able to consider the residue field and the value group as definable sets in a valued field (respectively, in a non-strict Cohen ring), it is necessary to switch from the language of valued fields (respectively, the language of rings) to a multisorted setting. Neither the results nor the proofs in this section are sensitive

to this technicality. An alternative approach is to widen the definition of stable embeddedness to also apply to interpretable sets, as discussed in [17].

In the appendix to [3], Chatzidakis and Hrushovski show stable embeddedness of a type-definable set is equivalent to an automorphism-lifting criterion in saturated models. This shows:

THEOREM 11.2. — *Let $A \models T_{\mathrm{pc}}(k, n)$. Then k_A is stably embedded in A.*

Proof. — By the Structure Theorem (Corollary 6.6), each automorphism of the residue field k_A lifts to an automorphism of the (non-strict) Cohen ring A. In particular this holds for sufficiently saturated models, and thus k_A is stably embedded as a pure field by [3, Appendix. Lemma 1]. □

Our aim is now to show that both residue field and value group are stably embedded in unramified henselian valued fields. Following the approach in [7, Lemma 3.1], we do this via our embedding lemma. More precisely, stable embeddedness follows from Proposition 10.1 in a straightforward manner since the type of an element of the residue field (respectively, the value group) over the residue field (resp., value group) of an elementary submodel determines the type of that element over the submodel:

THEOREM 11.3. — *Let (K, v) be an unramified henselian valued field. Then the value group vK and the residue field Kv are both stably embedded, as a pure ordered abelian group and as a pure field, respectively.*

Proof. — In light of Proposition 10.1, this is exactly the same argument as the proof of [7, Lemma 3.1]. □

As an immediate consequence, we get the following generalization of [25, Theorem 7.3] to the case of imperfect residue fields:

COROLLARY 11.4. — *Let (K, v) be an unramified henselian valued field. Then each subset of Kv^n which is $\mathfrak{L}_{\mathrm{vf}}$-definable in (K, v) is already $\mathfrak{L}_{\mathrm{ring}}$-definable in Kv.*

We now give an example that stable embeddedness of the residue field no longer holds for finite extensions of unramified henselian valued fields. A valued field (K, v) of mixed characteristic $(0, p)$ is **finitely ramified** if the interval $(0, v(p)]$ is finite. The following example shows that an analogue of Theorem 11.3 does not hold for all finitely ramified henselian valued fields.

Example 11.5. — Consider a field k of characteristic $p > 2$ with elements $\alpha_1, \alpha_2 \in k$ such that

    (i) there is an automorphism φ of k which maps α_1 to α_2, and

    (ii) α_1 and α_2 lie in different multiplicative cosets of $k^{\times 2}$.

Let (K, v) be an unramified henselian valued field with residue field k. We distinguish a representative $a_1 \in \mathrm{res}^{-1}(\alpha_1)$ of α_1, and let (L, w) be the extension of (K, v) given by adjoining a square-root b_1 of pa_1. Then (L, w) is henselian and finitely ramified, and $(L, w)/(K, v)$ is a quadratic extension with ramification degree $e = 2$ and inertia degree $f = 1$. It is also easy to see that L contains no element b_2 such that $b_2^2 = pa_2$, where a_2 is any representative of α_2. Suppose that Φ is an automorphism of (L, w) which lifts φ. Then

$$\mathrm{res}(\Phi(b_1)^2/p) = \mathrm{res}(\Phi(a_1)) = \varphi(\mathrm{res}(a_1)) = \alpha_2,$$

showing that $\Phi(b_1)$ is just such a non-existent element b_2 of L, which is a contradiction.

Note that there are elementary extensions (L^*, w^*) of (L, w) with an automorphism φ^* on the residue field $L^* w^*$ such that (i) and (ii) still hold. These can be obtained by adding a symbol for the automorphism φ to the language on the residue field before saturating. The argument above shows in fact that in none of these models, φ^* can be lifted to an automorphism of L^*.

Therefore the residue field $Lw = k$ is not stably embedded in (L, w). Finally, note that there are many such fields k; for example consider $k = \mathbb{F}_p(\alpha_1, \alpha_2)$ where α_1, α_2 are algebraically independent over $\mathbb{F}_p$.

Acknowledgements

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