CONFLUENTES MATHEMATICI

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Tome 5, nº 2 (2013), p. 79-85.

<http://cml.cedram.org/item?id=CML_2013__5_2_79_0>

Confluentes Math.
5, 2 (2013) 79-85

OCTONION MULTIPLICATION AND HEAWOOD'S MAP

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Abstract. In this note, the octonion multiplication table is recovered from a regular tesselation of the equilateral two timensional torus by seven hexagons, also known as Heawood's map.

Almost any article or book dealing with Cayley-Graves algebra $\mathbb{O}$ of octonions (to be recalled shortly) has a picture like the following

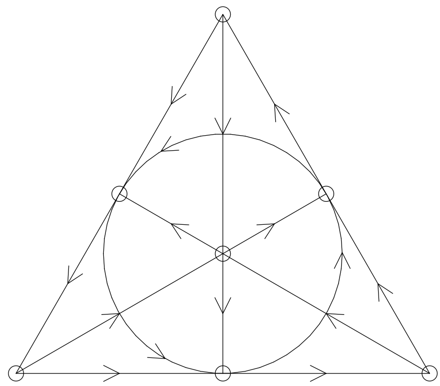


FIGURE 0.1.

representing the so-called 'Fano plane', which will be denoted by Π, together with some cyclic ordering on each of its 'lines'. The Fano plane is a set of seven points, in which seven three-point subsets called 'lines' are specified, such that any two points are contained in a unique line, and any two lines intersect in a unique point, giving a so-called (combinatorial) projective plane [8, 7]. In this cardinality, it is unique and may be seen as the projective plane over the field $\mathbb{F}_2$. In the above figure, the points of Π are represented by vertices, midpoints of sides, and barycenter of an equilateral triangle, and its 'lines' are either triples of aligned points (six sets) or the inscribed circle. Finally, the arrows on the figure indicate a cyclic ordering on each line (six arrows are omitted for clarity).

The recipe to obtain the multiplication table of the seven imaginary basic octonions e_a, $a \in \Pi$, from the above figure is then

- For all a, $e_a^2 = -1$.
- If a, b, c are the cyclically ordered points of a line, then

$$e_a e_b = e_c = -e_b e_a.$$

Finally one puts

$$\mathbb{O} = \mathbb{R}1 \oplus \bigoplus_{a \in \Pi} \mathbb{R}e_a \quad .$$

Math. classification: 17A35,05C10,05C25.

Recall that $\mathbb{O}$ is the largest finite-dimensional $\mathbb{R}$-algebra with unit possessing a multiplicative positive definite quadratic form $N : \mathbb{O} \to \mathbb{R}$ and is in particular a division algebra. The smaller ones are $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ (Hamilton quaternion algebra), and contrary to them $\mathbb{O}$ is not associative, but only *alternative*, meaning for example that the associator $\{x, y, z\} = (xy)z - x(yz)$ is skew-symmetric in (x, y, z). Any three e_a's with a's on a line form together with 1 a basis for an isomorphic copy of $\mathbb{H}$ inside $\mathbb{O}$. One can find many interesting aspects of this algebra in [1] or [4], and also on the web at `math.ucr.edu/home/baez/octonions/index.html` .

The purpose of this note is to present another graphical mnemonic for the same multiplication table, which avoids the need to remember the seemingly arbitrary orientations on figure 0.1:

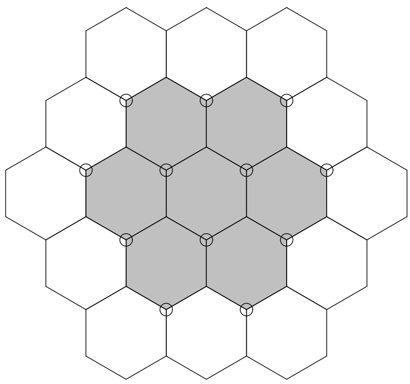


FIGURE 0.2.

This drawing is to be interpreted as follows. The union of the seven shaded hexagons is a fundamental domain for a complex lattice $\Lambda \subset \mathbb{C}$, generated by 'forward-right knight moves on an hexagonal chessboard'. More mathematically, one can take $\Lambda = (2 - \omega)\mathbb{Z}[\omega]$, an ideal of norm (index) 7 in the ring $\mathbb{Z}[\omega]$ of Eisenstein integers, $\omega = (-1 + \sqrt{-3})/2$. Then the seven elements $0, \pm 1, \pm\omega, \pm\omega^2$ of $\mathbb{Z}[\omega]$ form a complete set of representatives of $\mathbb{Z}[\omega]/\Lambda \simeq \mathbb{Z}/7\mathbb{Z}$, and the shaded hexagons are their Dirichlet-Voronoï cells relative to the lattice $\mathbb{Z}[\omega] \subset \mathbb{C}$ (points which are closer to the considered point than to any other lattice point).

Going to the quotient torus $\mathbb{C}/\Lambda$, one sees on it seven hexagonal regions, each of which is adjacent (along an edge) to the other six. Indeed this is obvious for the one with center 0, and translation invariance by $\mathbb{Z}[\omega]/\Lambda$ implies the claim. Dually, joining each hexagon's center $a \in \mathbb{Z}[\omega]/\Lambda$ to the centers $a \pm 1, a \pm \omega, a \pm \omega^2$ of its adjacent hexagons in $\mathbb{C}/\Lambda$, one obtains an embedding of the complete graph K_7 on seven vertices in the torus.

In particular seven colors are necessary to color maps on a torus, and it is not too difficult to show that they are also sufficient [8, p. 434]. This last fact and the above map on the torus were discovered around 1890 by Heawood [6], together with a gap in Kempe's 1879 proof of the four color theorem.

Now any edge $\{a, b\}$ of this K_7 may be oriented from a to b when $b - a \equiv 1, \omega, \omega^2 \bmod \Lambda$ (equivalently if $b - a$ is a square mod Λ), and from b to a otherwise. As $1 + \omega + \omega^2 = 0$, this defines a *cyclic orientation* on each set of three hexagons around a common vertex. The circled vertices in figure 0.2 are those where this orientation coincides with the complex (trigonometric) one (equivalently, those vertices where the edges form a "Y").

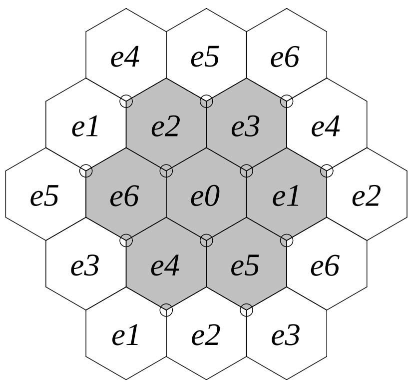
Remark 0.1. — In general, for q an odd prime power, the graph with set of vertices the finite field $\mathbb{F}_q$ and an edge from a to b iff $b-a$ is a nonzero square is called the Paley graph of order q. It is a tournament (i.e. an orientation of a complete graph) when -1 isn't a square in $\mathbb{F}^q$, e.g. $q=7$, and an unoriented regular graph (of degree $(q-1)/2$) otherwise.

The recipe to obtain the multiplication table of $\mathbb{O}$ is now : associate to each hexagon on the torus, *i.e.* to each vertex of K_7, or also to each element a of $\mathbb{Z}[\omega]/\Lambda \simeq \mathbb{Z}/7\mathbb{Z}$, an imaginary basic octonion e_a and declare that $e_a^2 = -1$ and $e_a e_b = e_c = -e_b e_a$ whenever a, b, c are cyclically ordered hexagons around a circled vertex.

To verify that the algebra thus defined is indeed $\mathbb{O}$ one can check that the complex oriented triangles in K_7, *i.e.* the triples of hexagons around a circled vertex, define the lines of a projective plane structure on the set of seven hexagons, and that the cyclic orientations of the lines are the same as in figure 0.1, with appropriate identifications.

Alternatively, one may observe that the ring isomorphism $\mathbb{Z}[\omega]/\Lambda \simeq \mathbb{Z}/7\mathbb{Z}$ sends ω mod Λ to 2 mod 7 and thus the multiplication rules are the same as in [4, p. 65] or [1, p. 150], namely — with indices modulo 7:

$$\begin{array}{rcccl} e_{n+1}e_{n+2} & = & e_{n+4} & = & -e_{n+2}e_{n+1} \\ e_{n+2}e_{n+4} & = & e_{n+1} & = & -e_{n+4}e_{n+2} \\ e_{n+4}e_{n+1} & = & e_{n+2} & = & -e_{n+1}e_{n+4}. \end{array}$$



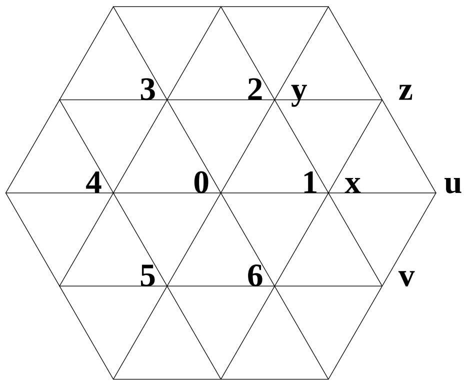
However, as this may seem a mere coincidence, we propose the following perhaps more conceptual statement.

Theorem 0.2. — *Up to isomorphism, there is a unique orientation of the graph K_7 for which oriented circuits of length three define a combinatorial triangulation of a surface (without boundary). This surface is a torus. Moreover, given an orientation of this torus, the triangles whose boundary orientation coincides with the graph orientation define a (Fano) projective plane structure on the set of vertices. The cyclic orderings of the lines in this plane coincide with those of figure 0.1.*

Proof. — Let us first consider a combinatorial triangulation of a surface S having K_7 as its 1-skeleton. It has 21 edges, each belonging to exactly two triangles, so there are t triangles, with $3t = 42$, *i.e.* $t = 14$. But this gives for the Euler characteristic $\chi(S) = 7 - 21 + 14 = 0$, hence S is a torus or a Klein bottle. We will exclude the second possibility (the proof is identical to that in [2, p. 157], and is given here for convenience).

Observe that around each vertex x of K_7, the triangles containing x together with the 'adjacency along an edge from x' relation, must form a single cycle of length 6, because S is a surface. Label 0 some vertex and $1, 2, \ldots, 6$ the others, in trigonometric order around 0. Then consider the universal cover $\widetilde{S}$ of S together with the lifted triangulation and labeling of vertices by $0, 1, 2, \ldots, 6$.

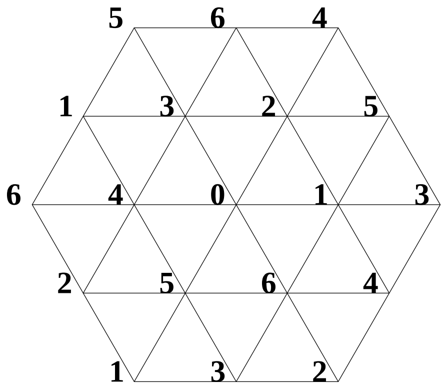
This gives a labeling of the vertices of the equilateral triangulation of the plane, such that all six labels different from a appear around each vertex labelled a, in a cycle that depends only on a (up to orientation, to allow for the Klein bottle possibility).



By the above choice, $1, 2, \ldots, 6$ appear in cyclic order around a 0-labelled vertex ('first shell'); then consider the labelling possibilities for the vertices at combinatorial distance 2 from this 0-vertex ('second shell') — there are 12 such vertices. It is easy to see for instance that the second shell vertex z adjacent to the first shell vertices x, y labelled respectively 1 and 2 can only be labelled 4 or 5, since labels $0, 1, 2, 3, 6$ are excluded — they already appear around x or y.

Assume that z is labelled 5 (the other choice is symmetrical). Denote by u the second shell vertex u adjacent to x, z and by v the second shell vertex adjacent to x and u (thus z, u, v appear in clockwise order around x). Then u can be labelled 3 or 4. If it were 4, then v is labelled 3, and around label 4 one would have sequences of labels $5 - 1 - 3$ (from vertex u), and also $5 - 0 - 3$ (from the first shell vertex labeled 4). This contradicts the fact that all labels must appear in a single cycle around each vertex of S. Thus u is labelled 3, and v is labelled 4. It is now fairly easy to see that the labelling of second shell vertices is uniquely determined (2,3,1,2,6,1,5,6,4 in clockwise order after vertex v), and that the deck transformations of $\widetilde{S}$ are translations by vectors in the lattice $\Lambda = (2 - \omega)\mathbb{Z}[\omega]$ of 'forward right knight moves'.

Note that in case z is labelled 4 instead, one obtains the mirror image lattice $\overline{\Lambda} = (2 - \omega^2)\mathbb{Z}[\omega]$ of 'forward *left* knight moves'.



In particular S is necessarily orientable (i.e. a torus), and up to (simplicial) isomorphism, there is only one triangulation of a surface having K_7 as its 1-skeleton.

To finish the proof of the first statement in the theorem, observe that the resulting 14 triangles map is 2-colorable (equivalently, the dual graph is bipartite), for instance coloring in black triangles 'pointing up' in the universal covering plane. Orienting the edges of K_7 as boundaries of black triangles (for some orientation of the torus) gives a solution to the first statement, and the uniqueness (up to isomorphism) results from that of the triangulation and from consideration of $-\mathrm{Id}$, which exchanges the two colors.

Checking that coherently oriented (alias black) triangles define the lines of a projective plane is easy[1]: they obviously form a set of seven 'lines', 3-element subsets of a 7-element set of 'points', such that any point is on exactly 3 lines and any two points are on a unique line. A simple counting argument shows that any two lines intersect then in a unique point.

To verify that the resulting cyclic orderings of lines correspond to those represented in figure 0.1, one only has to see (by uniqueness) that the latter verify the triangulation property of the theorem. This is not difficult, taking symmetries into account. Note that to obtain an oriented K_7, one must add six oriented edges to the graph of figure 0.1, namely those "closing" the three sides and three medians. Then the reader will quickly check that any oriented edge is contained in exactly two oriented circuits of length 3, and that those oriented 'triangles' containing any vertex x form a single cycle with respect to the relation of adjacency along an edge from x. □

Remarks 0.3. — (1) The link between the K_7 graph on the torus and the Fano projective plane Π has yet another manifestation. Namely, its dual graph, union of the boundaries of the seven hexagons map, is the incidence graph of Π : it is the bipartite graph with vertices the points and lines of Π, with an edge for each membership, also known as the Heawood graph. To see this, it is enough to observe that on figure 0.2, the downward translation of length the side of one hexagon maps the hexagons' centers (vertices of Π) to circled vertices and circled vertices (lines of Π) to uncircled vertices, thus identifiying Heawood's graph with the vertex graph of the hexagons.

[1] Although it already results from the octonion multiplication table given before the theorem and from the classical figure 1, we prefer to give an independent check, since it is easy.

(2) Since it is easy to show that any embedding of K_7 in a closed surface of Euler characteristic 0 has triangular faces, the above proof shows that K_7 embeds uniquely (up to homeomorphism) in the torus, and also recovers the fact that it doesn't embed in the Klein bottle.

(3) The automorphism group of the oriented K_7 in the theorem is easily seen to be isomorphic to $F_{21} = \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{F}_7 \rtimes \mathbb{F}_7^{*2}$, for example via its realization as the (oriented) Paley graph on $\mathbb{F}_7$. This exhibits F_{21} as a finite subgroup of the compact exceptional Lie group $G_2 = \mathrm{Aut}(\mathbb{O})$, and of course also as a subgroup of the automorphism group of the Fano plane $\mathrm{Aut}(\Pi) = GL_3(\mathbb{F}_2) \simeq PSL_2(\mathbb{F}_7)$ [2], the simple group of order $21.8 = 168$.

It is then only natural to ask if it is possible to find a finite subgroup of G_2 "realizing" in some way the whole of $\mathrm{Aut}(\Pi)$. This turns out to be possible, namely as the automorphism group of the loop[3] $O_{16} = \{\pm e_x \mid x \in \mathbb{F}_2^3 = \Pi \cup 0\} \subset \mathbb{O}^*$, with $e_0 = 1$ (see [5, sect. 4]). First note that any element of $\mathrm{Aut}(O_{16})$ preserves the center $\{\pm 1\}$, hence induces an automorphism of the quotient $O_{16}/\{\pm 1\} = \mathbb{F}_2^3$. By simplicity of $GL_3(\mathbb{F}_2)$, and since one already has $F_{21} \subset \mathrm{Aut}(O_{16})$, it is sufficient to find one element of $\mathrm{Aut}(O_{16})$ of the form $e_x \mapsto \pm e_{tx}$ with t of order 2 in $GL_3(\mathbb{F}_2)$, which is easy. The catch is that one obtains a group extension

$$0 \to (\mathbb{F}_2^3)^\vee \to \mathrm{Aut}(O_{16}) \to GL_3(\mathbb{F}_2) \to 1$$

(with the natural action of the quotient on the kernel) which has *no section*, i.e. is not the obvious semi-direct product. The kernel is the group of characters $\mathbb{F}_2^3 \to \{\pm 1\}$, i.e. more concretely the group of sign changes $e_x \mapsto (-1)^{\ell(x)} e_x$, with ℓ any linear form on $\mathbb{F}_2^3$.

Such non-split extensions $GL_n(\mathbb{F}_q)$ by its natural module exist only for $q = 2$ and $n = 3, 4, 5$, the largest being known as the Dempwolff group, see `http://en.wikipedia.org/wiki/Dempwolff_group`.

Acknowledgements

I would like to thank an anonymous referee for his or her very thorough reading and useful remarks.

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[2] See [3] for a nice account of this isomorphism, and [9] for a more sophisticated one.

[3] Namely, a set with a composition law which has a unit and where left and right multiplications are bijections.

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Manuscript received September 11, 2012,
revised May 30, 2013,
accepted June 20, 2013.

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