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NON-LOCAL APPROXIMATIONS OF THE GRADIENT

HAIM BREZIS AND PETRU MIRONESCU

Abstract. We revisit the proofs of a few basic results concerning non-local approximations of the gradient. A typical such result asserts that, if (ρ_ε) is a radial approximation to the identity in $\mathbb{R}^N$ and u belongs to a homogeneous Sobolev space $\dot{W}^{1,p}$, then

$$V_\varepsilon(x) := N \int_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h)\, dh,\ x \in \mathbb{R}^N,$$

converges in L^p to the distributional gradient ∇u as $\varepsilon \to 0$.

We highlight the crucial role played by the representation formula $V_\varepsilon = (\nabla u) * F_\varepsilon$, where F_ε is an approximation to the identity defined *via* ρ_ε. This formula allows to unify the proofs of a significant number of results in the literature, by reducing them to standard properties of the approximations to the identity.

We also highlight the effectiveness of a symmetric non-local integration by parts formula.

Relaxations of the assumptions on u and ρ_ε, allowing, e.g., heavy tails kernels or a distributional definition of V_ε, are also discussed. In particular, we show that heavy tails kernels may be treated as perturbations of approximations to the identity.

1. A representation formula and applications

Let $(\rho_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family functions on $\mathbb{R}^N$ such that:

$$\rho_\varepsilon \text{ is non-negative, integrable, radial, } \forall\ \varepsilon, \tag{1.1}$$

$$\int_{\mathbb{R}^N} \rho_{\varepsilon=1},\ \forall\ \varepsilon, \tag{1.2}$$

$$\lim_{\varepsilon \to 0} \int_{|h|>\delta} \rho_\varepsilon(h)\, dh = 0,\ \forall\ \delta > 0. \tag{1.3}$$

Following Mengesha and Spector [7] (with roots in Bourgain, Brezis, and Mironescu [1], Gilboa and Osher [6], Du, Gunzburger, Lehouck, and Zhou [5]; see also a detailed list of references in [7, p. 254]), we set, for any measurable function $u \in L^1_{loc}(\mathbb{R}^N)$, and assuming that the integral below exists,

$$\begin{aligned} V_\varepsilon(x) &= V_{\varepsilon,u}(x) \\ &= V_{\varepsilon,u,\rho_\varepsilon}(x) := N \int_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h)\, dh, x \in \mathbb{R}^N. \end{aligned} \tag{1.4}$$

V_ε may be seen as a non-local approximation of the gradient. Indeed (see Remark 1.13), (i) when u is C^1 and bounded, we have the pointwise convergence $V_\varepsilon(x) \to \nabla u(x)$ as $\varepsilon \to 0$, $\forall\ x \in \mathbb{R}^N$; (ii) when u is C^1 and compactly supported, we have $V_\varepsilon \to \nabla u$ uniformly.

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In what follows, we revisit the proofs of a few results establishing the validity of the convergence

$$V_\varepsilon \to Du \text{ as } \varepsilon \to 0, \tag{1.5}$$

in various functional settings. Many of these results were originally obtained, in slightly different forms, in [7].

Before presenting the main results and methods, we make some easy observations concerning the existence of V_ε. Set

$$W_\varepsilon(x) = W_{\varepsilon,u}(x) = W_{\varepsilon,u,\rho_\varepsilon}(x) := \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|} \rho_\varepsilon(h)\, dh,\ x \in \mathbb{R}^N.$$

Clearly, the following holds.

LEMMA 1.1. — *Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Then V_ε is well-defined a.e. and is measurable.*

Moreover, we have $|V_\varepsilon| \leqslant N\, W_\varepsilon$ a.e., and thus $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

Remark 1.2. — In the above statements, the condition $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$ seems constraining. However, under the following assumption:

$$\text{for every } \varepsilon, \text{there exist some } \delta_\varepsilon, R_\varepsilon > 0 \text{ such that } \rho_\varepsilon(h) = 0 \text{ if } |h| < \delta_\varepsilon \text{ or if } |h| > R_\varepsilon, \tag{1.6}$$

we have $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$, $\forall\ u \in L^1_{loc}(\mathbb{R}^N)$.

This is especially relevant for Propositions 2.2, 2.3, 2.5, 2.6, and 5.1 below.

We next present a sufficient condition for having $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ (and thus, by Lemma 1.1, $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$). For $1 \leqslant p < \infty$, set

$$I_{\varepsilon,p} = I_{\varepsilon,p,u} = I_{\varepsilon,p,u,\rho_\varepsilon} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p} \rho_\varepsilon(h)\, dxdh.$$

LEMMA 1.3. — *Assume* (1.2). *Let $1 \leqslant p < \infty$ and $u \in L^1_{loc}(\mathbb{R}^N)$. Then $\|W_\varepsilon\|^p_{L^p(\mathbb{R}^N)} \leqslant I_{\varepsilon,p}$.*

Consequently, if $I_{\varepsilon,p} < \infty$, then V_ε is well-defined a.e., measurable, and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leqslant N\, [I_{\varepsilon,p}]^{1/p}$.

In the above and in what follows, the L^p-norms of vector fields $F : \mathbb{R}^N \to \mathbb{R}^N$ are computed with respect to the Euclidean norm $|\ |$, i.e.,

$$\|F\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |F(x)|^p\, dx.$$

Similarly, the mass of a measure $F \in \mathscr{M}(\mathbb{R}^N;\mathbb{R}^N)$ is computed with respect to the Euclidean norm, i.e.,

$$\|F\|_{\mathscr{M}(\mathbb{R}^N)} = \sup\left\{ \sum_{j=1}^N \int_{\mathbb{R}^N} \zeta_j\, dF_j;\ \zeta \in C^\infty_c\left(\mathbb{R}^N;\mathbb{R}^N\right),\ |\zeta(x)| \leqslant 1,\ \forall\ x \right\}.$$

Proof of Lemma 1.3. — The conclusion follows by integrating in x the inequality

$$[W_\varepsilon(x)]^p \leqslant \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p}\rho_\varepsilon(h)\,dh \left(\int_{\mathbb{R}^N} \rho_\varepsilon(h)\,dh\right)^{p-1}$$
$$= \int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p}\rho_\varepsilon(h)\,dh,\ \forall\ x \in \mathbb{R}^N.$$

□

We now recall a few sufficient conditions for having $I_{\varepsilon,p} < \infty$. Set

$$\dot{W}^{1,p} := \left\{u \in \mathscr{D}'\left(\mathbb{R}^N\right);\ Du \in L^p\left(\mathbb{R}^N\right)\right\}$$
$$= \left\{u \in L^1_{loc}\left(\mathbb{R}^N\right);\ Du \in L^p\left(\mathbb{R}^N\right)\right\}, 1 \leqslant p < \infty,$$
$$\dot{BV} := \left\{u \in \mathscr{D}'\left(\mathbb{R}^N\right);\ Du \in \mathscr{M}\left(\mathbb{R}^N\right)\right\}$$
$$= \left\{u \in L^1_{loc}\left(\mathbb{R}^N\right);\ Du \in \mathscr{M}\left(\mathbb{R}^N\right)\right\}.$$

In what follows, for $u \in \dot{W}^{1,p}$, we denote the distributional gradient ∇u.
Let $K_{p,N} := \fint_{\mathbb{S}^{N-1}} |h_j|^p\, d\sigma(h)$ (which does not depend on $j \in [\![1, N]\!]$).
We have the following

Lemma 1.4 ([1]). — *Assume* (1.1).

(1) *Let* $1 \leqslant p < \infty$ *and* $u \in \dot{W}^{1,p}$. *Then* $I_{\varepsilon,p} \leqslant K_{p,N}\, \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\, \|\nabla u\|^p_{L^p(\mathbb{R}^N)}$.
In particular, $V_\varepsilon \in L^p(\mathbb{R}^N)$ *and*

$$\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leqslant N\, [K_{p,N}]^{1/p}\, \|\rho_\varepsilon\|^{1/p}_{L^1(\mathbb{R}^N)}\, \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

(2) *Let* $u \in \dot{BV}$. *Then* $I_{\varepsilon,1} \leqslant K_{1,N}\, \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\, \|Du\|_{\mathscr{M}(\mathbb{R}^N)}$.
In particular, $V_\varepsilon \in L^1(\mathbb{R}^N)$ *and*

$$\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leqslant N\, K_{1,N}\, \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\, \|Du\|_{\mathscr{M}(\mathbb{R}^N)}.$$

We next present a crucial identity that illuminates the validity of (1.5)*:* (1.10), *and its avatar* (1.11). Although (1.10) was probably known to experts (it is implicit in [7, proof of Lemma 3.3] and related to several identities in [5]), its intimate connection to (1.5) seems to have remained relatively unnoticed.

Assume (1.1). Let $f_\varepsilon : (0,\infty) \to [0,\infty)$ be a measurable function such that $\rho_\varepsilon(x) = f_\varepsilon(|x|)$ for a.e. $x \in \mathbb{R}^N$. Set

$$F_\varepsilon(h) := N \int_{|h|}^{\infty} \frac{f_\varepsilon(t)}{t}\, dt,\ \forall\ h \in \mathbb{R}^N \setminus \{0\}. \tag{1.7}$$

Let us note that

$$\|F_\varepsilon\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} F_\varepsilon = N\, |\mathbb{S}^{N-1}| \int_0^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t}\, dtdr$$
$$= |\mathbb{S}^{N-1}| \int_0^\infty t^{N-1}\, f_\varepsilon(t)\, dt = \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, dh = \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}, \tag{1.8}$$

so that, in particular,

$$F_\varepsilon \in L^1(\mathbb{R}^N). \tag{1.9}$$

Lemma 1.5. — *Assume* (1.1). *Let* F_ε *be as in* (1.7).

(1) *Let* $1 \leqslant p < \infty$ *and* $u \in \dot{W}^{1,p}$. *Then*

$$V_\varepsilon = (\nabla u) * F_\varepsilon \text{ a.e.} \tag{1.10}$$

(2) *Let* $u \in \dot{BV}$. *Then*

$$V_\varepsilon = (Du) * F_\varepsilon \text{ a.e.} \tag{1.11}$$

Proof. —

Step 1. Proof of (1.10) *when* $u \in C^\infty(\mathbb{R}^N)$ *and* $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$. — In this case, we actually prove that

$$V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x), \ \forall \ x \in \mathbb{R}^N.$$

For this purpose, we note that

$$F_\varepsilon \text{ is compactly supported,} \tag{1.12}$$

$$\nabla F_\varepsilon(h) = -N \frac{h}{|h|^2} \rho_\varepsilon(h), \ \forall \ h \in \mathbb{R}^N \setminus \{0\}, \tag{1.13}$$

$$F_\varepsilon(h) = O(|\ln |h||) \text{ as } h \to 0. \tag{1.14}$$

Using (1.12)–(1.14), we find, via an integration by parts, that

$$\begin{aligned} V_\varepsilon(x) &= -\int_{\mathbb{R}^N} [u(x+h) - u(x)] \, \nabla F_\varepsilon(h) \, dh \\ &= -\lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus \overline{B}(0,\delta)} [u(x+h) - u(x)] \, \nabla F_\varepsilon(h) \, dh \\ &= \int_{\mathbb{R}^N} \nabla u(x+h) \, F_\varepsilon(h) \, dh = \int_{\mathbb{R}^N} \nabla u(x-h) \, F_\varepsilon(h) \, dh \\ &= [(\nabla u) * F_\varepsilon](x), \ \forall \ x \in \mathbb{R}^N. \end{aligned}$$

Step 2. Proof of (1.10) *and* (1.11) *when* $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. – Let η be a radial non-increasing normalized bump function. By Step 1, we have

$$V_{\varepsilon, u*\eta_\delta}(x) = (\nabla(u * \eta_\delta)) * F_\varepsilon(x) = (Du) * (F_\varepsilon * \eta_\delta)(x), \ \forall \ x \in \mathbb{R}^N. \tag{1.15}$$

On the one hand, as $\delta \to 0$, the right-hand side of (1.15) converges (possibly along a subsequence) a.e. to $(Du) * F_\varepsilon(x)$. (This follows by combining the Young inequality with the fact that $F_\varepsilon * \eta_\delta \to F_\varepsilon$ in L^1.)

In order to obtain (1.10), respectively (1.11), it suffices to prove that

$$V_{\varepsilon, u*\eta_\delta, \rho_\varepsilon}(x) \to V_{\varepsilon, u, \rho_\varepsilon}(x) \text{ as } \delta \to 0 \text{ for a.e. } x \in \mathbb{R}^N. \tag{1.16}$$

Property (1.16) is obtained *via* dominated convergence, using the standard inequality

$$|u * \eta_\delta(y)| \leqslant \mathscr{M}_1 u(y), \forall \ 0 < \delta < 1, \ \forall \ y \in \mathbb{R}^N, \tag{1.17}$$

(see, e.g., [13, eq (17), p. 57]), where $\mathscr{M}_1 u$ is the centered truncated maximal function of u,

$$\mathscr{M}_1 u(x) := \sup \left\{ \fint_{B_r(x)} |u|;\ 0 < r \leqslant 1 \right\}.$$

(Here, we use the fact that η is radial, non-increasing, and supported in the unit ball.) Using (1.17) and the extra assumptions on ρ_ε, we obtain the domination

$$\frac{|u * \eta_\delta(x+h) - u * \eta_\delta(x)|}{|h|} \rho_\varepsilon(h) \leqslant [\mathscr{M}_1 u(x+h) + \mathscr{M}_1 u(x)]\, g(h), \forall\ 0 < \delta < 1, \quad (1.18)$$

with $g(h) := \rho_\varepsilon(h)/|h|$ bounded and compactly supported. The right-hand side of (1.18) is in $L^1(\mathbb{R}^N)$ since $\mathscr{M}_1 u \in L^1_{loc}(\mathbb{R}^N)$ (and thus, in particular, $\mathscr{M}_1 u$ is finite a.e.). The latter property follows by combining the Sobolev embeddings $\dot{W}^{1,p}$, $\dot{BV} \hookrightarrow L^{N/(N-1)}_{loc}(\mathbb{R}^N)$ with the fact that, by the maximal function theorem, we have $\mathscr{M}_1 u \in L^r_{loc}(\mathbb{R}^N)$ when $u \in L^r_{loc}(\mathbb{R}^N)$ for some $r > 1$. We obtain that the convergence in (1.16) holds on the full measure set

$$\left\{ x \in \mathbb{R}^N;\ \mathscr{M}_1 u(x) < \infty \text{ and } x \text{ is a Lebesgue point of } u \right\}.$$

Step 3. Proof of (1.10) *and* (1.11) *in the general case.* — For fixed ε, we approximate ρ_ε in L^1 with a sequence $(\rho_{\varepsilon,j})_j$ of kernels $\rho_{\varepsilon,j} \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ satisfying (1.1). By Step 2, the corresponding associated kernels $F_{\varepsilon,j}$ satisfy

$$V_{\varepsilon,u,\rho_{\varepsilon,j}} = (Du) * F_{\varepsilon,j} \text{ a.e.} \quad (1.19)$$

Let us note that $F_{\varepsilon,j} \to F_\varepsilon$ in $L^1(\mathbb{R}^N)$ as $j \to \infty$. (This follows from a straightforward variant of (1.8).)

We obtain (1.10), respectively (1.11), by letting $j \to \infty$ in (1.19). Passing to the limits is justified, on the left-hand side, by Lemma 1.4, and, on the right-hand side, by the Young inequality combined with the fact that $F_{\varepsilon,j} \to F_\varepsilon$ in $L^1(\mathbb{R}^N)$. □

Using Lemma 1.5, (1.8), and the Young inequality, we obtain the following

Lemma 1.6. — *Assume* (1.1)–(1.2).

(1) *Let* $1 \leqslant p < \infty$ *and* $u \in \dot{W}^{1,p}$. *Then* $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^p(\mathbb{R}^N)}$.
(2) *Let* $u \in \dot{BV}$. *Then* $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leqslant \|Du\|_{\mathscr{M}(\mathbb{R}^N)}$.

This is an improvement of Lemma 1.4, since $N\,[K_{p,N}]^{1/p} > 1$ when $N \geqslant 2$. Indeed, the Jensen inequality yields

$$K_{p,N} \geqslant \left(\fint_{\mathbb{S}^{N-1}} |h_j|\, d\sigma(h) \right)^p = \left(\frac{1}{N} \fint_{\mathbb{S}^{N-1}} \sum_{k=1}^N |h_k|\, d\sigma(h) \right)^p > \left(\frac{1}{N} \fint_{\mathbb{S}^{N-1}} d\sigma(h) \right)^p = \frac{1}{N^p}.$$

We next present two direct consequences of Lemma 1.5, originally obtained, with different arguments, in [7].

Proposition 1.7 ([7, Theorem 1.1 (b)]). — *Assume* (1.1)–(1.3). *Let* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. *Then*

$$V_\varepsilon \to \nabla u \text{ in } L^p\left(\mathbb{R}^N\right) \text{ as } \varepsilon \to 0.$$

Proposition 1.8 ([7, Theorem 1.2]). — *Assume* (1.1)–(1.3). *Let* $u \in \dot{BV}$. *Then*

$$V_\varepsilon \rightharpoonup Du \text{ $*$-weakly in } \mathscr{M}\left(\mathbb{R}^N\right) \text{ as } \varepsilon \to 0$$

and

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)}.$$

Proof of Propositions 1.7 and 1.8. — By Lemma 1.9 below, (F_ε) is an approximation to the identity. We conclude by combining this fact with Lemma 1.5. □

Lemma 1.9. — *Under the assumptions* (1.1)–(1.3), (F_ε) *is an approximation to the identity.*

Proof. — If $\delta > 0$ is fixed, then (1.7) and (1.3) yield

$$\begin{aligned}\int_{|h|>\delta} F_\varepsilon(h)\,dh &= N\left|\mathbb{S}^{N-1}\right| \int_\delta^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t}\,dtdr \\ &= N\left|\mathbb{S}^{N-1}\right| \int_\delta^\infty \int_\delta^t r^{N-1}\,dr \frac{f_\varepsilon(t)}{t}\,dt \\ &\leqslant \left|\mathbb{S}^{N-1}\right| \int_\delta^\infty t^{N-1} f_\varepsilon(t)\,dt \\ &= \int_{|h|>\delta} \rho_\varepsilon(h)\,dh \to 0 \text{ as } \varepsilon \to 0.\end{aligned} \tag{1.20}$$

The conclusion of the lemma follows from (1.8) and (1.20). □

We next present two a.e. versions of the above results.

Proposition 1.10. — *Assume* (1.1)–(1.3). *Let* $u \in \dot{W}^{1,p}$. *Then, for a.e.* $x \in \mathbb{R}^N$, *we have* $V_\varepsilon(x) \to \nabla u(x)$ *as* $\varepsilon \to 0$.

Proposition 1.11. — *Assume* (1.1)–(1.3). *Let* $u \in \dot{BV}$. *Then, for a.e.* $x \in \mathbb{R}^N$, *we have* $V_\varepsilon(x) \to \nabla^{ac} u(x)$ *as* $\varepsilon \to 0$.

These results clearly follow from Lemmas 1.5 and 1.9 and the following well-known measure-theoretical result (see, e.g., [12, Chapter III, Section 2.2, Theorem 2] for the first item, and the discussion in [12, Chapter III, Section 4.1] for the second one).

Lemma 1.12. — *Let* (F_ε) *be a an approximation to the identity, with* F_ε *radial and non-increasing. Then*

(1) *For every* $1 \leqslant p < \infty$ *and* $G \in L^p(\mathbb{R}^N)$, *we have* $G * F_\varepsilon(x) \to G(x)$ *at each Lebesgue point of* G.

(2) *For every finite Borel measure ν singular with respect to the Lebesgue measure in $\mathbb{R}^N$, we have $\nu * F_\varepsilon \to 0$ a.e.*

Proof of Lemma 1.12. — The assumptions on F_ε imply that there exist (unique) non-negative Borel measures μ_ε on $(0,\infty)$ such that

$$F_\varepsilon(x) = \mu_\varepsilon((|x|,\infty)), \text{ for a.e. } x \in \mathbb{R}^N, \tag{1.21}$$

$$\frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \, d\mu_\varepsilon(t) = 1, \ \forall\ \varepsilon, \tag{1.22}$$

$$\lim_{\varepsilon \to 0} \frac{|\mathbb{S}^{N-1}|}{N} \int_\delta^\infty t^N \, d\mu_\varepsilon(t) = 1, \ \forall\ \delta > 0. \tag{1.23}$$

Proof of item (1). — We have

$$\begin{aligned}
&\left|G * F_\varepsilon(x) - G(x)\right| \\
&\quad = \left| \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} [G(x - r\omega) - G(x)] \, ds(\omega) \, \mu_\varepsilon((r,\infty)) \, dr \right| \\
&\quad \leqslant \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| \, ds(\omega) \, \mu_\varepsilon((r,\infty)) \, dr \\
&\quad = \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| \, ds(\omega) \int_r^\infty d\mu_\varepsilon(t) \, dr \\
&\quad = \int_0^\infty \int_0^t r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| \, ds(\omega) \, dr \, d\mu_\varepsilon(t) \qquad (1.24) \\
&\quad = \int_0^\infty \int_{B(x,t)} |G(y) - G(x)| \, dy \, d\mu_\varepsilon(t) \\
&\quad = \int_0^\infty |B(x,t)| \fint_{B(x,t)} |G(y) - G(x)| \, dy \, d\mu_\varepsilon(t) \\
&\quad = \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \fint_{B(x,t)} |G(y) - G(x)| \, dy \, d\mu_\varepsilon(t).
\end{aligned}$$

We complete the proof by combining (1.22)–(1.24) with the fact that

$$\lim_{t \to 0} \fint_{B(x,t)} |G(y) - G(x)| \, dy = 0 \text{ at each Lebesgue point } x \text{ of } G$$

and the straightforward inequality

$$\fint_{B(x,t)} |G(y)-G(x)|\,dy \leqslant \frac{1}{|B(x,t)|^{1/p}}\|G\|_{L^p(\mathbb{R}^N)} + |G(x)| \\ \leqslant \frac{1}{|B(x,\delta)|^{1/p}}\|G\|_{L^p(\mathbb{R}^N)} + |G(x)|,\ \forall\ t \geqslant \delta.$$

Proof of item (2). — We have

$$\begin{aligned}\nu * F_\varepsilon(x) \\ &= \int_{\mathbb{R}^N} \mu_\varepsilon((|x-y|,\infty))\,d\nu(y) = \int_{\mathbb{R}^N}\int_{|x-y|}^{\infty} d\mu_\varepsilon(t)\,d\nu(y) \\ &= \int_0^{\infty}\int_{|x-y|<t} d\nu(y)\,d\mu_\varepsilon(t) = \frac{|\mathbb{S}^{N-1}|}{N}\int_0^{\infty} t^N\,\frac{\nu(B(x,t))}{|B(x,t)|}\,d\mu_\varepsilon(t).\end{aligned} \tag{1.25}$$

We complete the proof combining (1.22), (1.23), and (1.25) with the fact that (by the Lebesgue–Besicovitch differentiation theorem) we have

$$\lim_{t\to 0}\frac{\nu(B(x,t))}{|B(x,t)|} = 0 \text{ for a.e. } x \in \mathbb{R}^N,$$

and the upper bound

$$\frac{\nu(B(x,t))}{|B(x,t)|} \leqslant \frac{\nu(\mathbb{R}^N)}{|B(x,\delta)|},\ \forall\ t \geqslant \delta. \qquad \square$$

For other results in the spirit of Propositions 1.10 and 1.11, see Spector [11, Theorem 1.2] and Brezis and Nguyen [2, Theorems 1 and 2].

Remark 1.13. — Here are two additional quick consequences of the fact that, under the assumptions (1.1)–(1.3), (F_ε) is an approximation to the identity. It is straightforward that the pointwise equality $V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x)$, $\forall\ x \in \mathbb{R}^N$, holds if $u \in C^1_c(\mathbb{R}^N)$, and therefore for such u we have $V_\varepsilon \to \nabla u$ uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$. Similarly, this equality holds when $u \in (C^1 \cap L^\infty)(\mathbb{R}^N)$, and in this case we have $V_\varepsilon(x) \to \nabla u(x)$, as $\varepsilon \to 0$, $\forall\ x \in \mathbb{R}^N$.

2. An integration by parts formula and applications

The representation formula (1.10) naturally leads to the following formal calculation, with $\zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$, $\check{f}(x) := f(-x)$, $\forall\ x \in \mathbb{R}^N$, and $\{e_j\}_{1\leqslant j\leqslant N}$ the canonical basis of $\mathbb{R}^N$:

$$\begin{aligned}\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta &= \sum_j \int_{\mathbb{R}^N} [V_{\varepsilon,u}\cdot e_j]\,\zeta_j = \sum_j [V_{\varepsilon,u}\cdot e_j] * \check{\zeta}_j(0)\\ &= \sum_j [(\partial_j u) * F_\varepsilon] * \check{\zeta}_j(0)\\ &= \sum_j u * \left[F_\varepsilon * \partial_j \check{\zeta}_j\right](0) = \sum_j u * V_{\varepsilon,\check{\zeta}_j}\cdot e_j(0)\\ &= \int_{\mathbb{R}^N} u(x) \sum_j \left[V_{\varepsilon,\check{\zeta}_j}\cdot e_j\right](-x).\end{aligned} \tag{2.1}$$

Combining (2.1) with the (formal) identity $V_{\varepsilon,\check{f}}(-x) = -V_{\varepsilon,f}(x)$, we obtain the formal identity

$$\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta = -\sum_{j=1}^N \int_{\mathbb{R}^N} u(x)\,\left[V_{\varepsilon,\zeta_j}(x)\cdot e_j\right]\,dx, \tag{2.2}$$

and its more symmetric avatar

$$\begin{aligned}&\int_{\mathbb{R}^N} [V_{\varepsilon,u}\cdot e_j]\,\psi\\ &\qquad = -\int_{\mathbb{R}^N} u(x)\,[V_{\varepsilon,\psi}(x)\cdot e_j]\,dx, \forall\ j,\ \forall\ \psi\in C_c^\infty\left(\mathbb{R}^N;\mathbb{R}\right).\end{aligned} \tag{2.3}$$

Similar "non-local integration by parts" identities were known in the literature (see, e.g., [5], [7, Theorem 1.4], and, in a slightly different setting, Šilhavý [10, Section 6]). As we will see below, (2.2) holds under mild assumptions on u (this is to be compared with the more restrictive assumptions in Lemma 1.5). The importance of such identities is that they provide a first direction for generalizing the results in Section 1, consisting of weakening the *assumption* $u\in \dot{W}^{1,p}$ (respectively $u\in \dot{BV}$), widely used in Section 1, to a reasonable one allowing V_ε to be well-defined a.e. and to obtain the *property* $u\in\dot{W}^{1,p}$ (respectively $u\in\dot{BV}$) *as a conclusion*.

We first formalize the validity of (2.2).

Lemma 2.1. — *Let $\varepsilon>0$ be fixed. Assume* (1.1). *Let $u\in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon\in L^1_{loc}(\mathbb{R}^N)$.*

(1) *If $u\in (L^1+L^\infty)(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta = -\sum_{j=1}^N \int_{\mathbb{R}^N} u(x)\,\left[V_{\varepsilon,\zeta_j}(x)\cdot e_j\right]\,dx,\ \forall\ \zeta\in C_c^\infty\left(\mathbb{R}^N;\mathbb{R}^N\right). \tag{2.4}$$

(2) *If ρ_ε is compactly supported, then* (2.4) *holds.*

Proof. — We first note the following equalities, valid (thanks to the Fubini theorem applied to the first line) for every $\zeta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$:

$$
\begin{aligned}
\int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h\cdot\zeta(x)}{|h|} \rho_\varepsilon(h)\, dh \right) dx \\
&= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h\cdot\zeta(x)}{|h|} \rho_\varepsilon(h)\, dx \right) dh \\
&= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h\cdot[\zeta(x-h)-\zeta(x)]}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh \\
&= -N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h\cdot[\zeta(x+h)-\zeta(x)]}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh.
\end{aligned}
\tag{2.5}
$$

We next claim that we may apply the Fubini theorem to the last integral in (2.5). By linearity, in item (1) we may assume that either $u \in L^1(\mathbb{R}^N)$ or $u \in L^\infty(\mathbb{R}^N)$.

Proof of the claim when $u \in L^1(\mathbb{R}^N)$. — In this case, we have

$$
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{\big|h\cdot[\zeta(x+h)-\zeta(x)]\big|}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh \\
\leqslant \|\nabla\zeta\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.
$$

Proof of the claim when $u \in L^\infty(\mathbb{R}^N)$. — By Lemma 1.4 (1), we have

$$
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{\big|h\cdot[\zeta(x+h)-\zeta(x)]\big|}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh \\
\leqslant K(1,N) \|u\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N \|\nabla\zeta_j\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.
$$

Proof of the claim when ρ_ε is compactly supported. — Let $r, R > 0$ be such that $\operatorname{supp}\zeta \subset B(0,r)$ and $\operatorname{supp}\rho_\varepsilon \subset B(0,R)$. Set $v := u\chi_{B(0,r+R)} \in L^1(\mathbb{R}^N)$. Then

$$
u(x) \frac{h\cdot[\zeta(x+h)-\zeta(x)]}{|h|^2} \rho_\varepsilon(h) \\
= v(x) \frac{h\cdot[\zeta(x+h)-\zeta(x)]}{|h|^2} \rho_\varepsilon(h), \ \forall\ x, h \in \mathbb{R}^N,
$$

and we then argue as in the case where $u \in L^1(\mathbb{R}^N)$.

Applying the Fubini theorem in (2.5), we find that

$$\begin{aligned}\int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h)\, dh \right) dx \\ &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \sum_{j=1}^{N} \frac{\zeta_j(x+h) - \zeta_j(x)}{|h|} \frac{h_j}{|h|} \rho_\varepsilon(h)\, dh \right) dx \\ &= -\sum_{j=1}^{N} \int_{\mathbb{R}^N} u(x) \left[V_{\varepsilon,\zeta_j}(x) \cdot e_j \right] dx,\end{aligned}$$

so that (2.4) holds. □

Here are two quick consequences of (2.4), in the spirit of [7, Theorems 1.5 and 1.6].

Proposition 2.2. — *Assume* (1.1)–(1.3)*. Let* $1 < p < \infty$*. Let* $u \in L^1_{loc}(\mathbb{R}^N)$ *be such that* $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall\ \varepsilon$.

(1) *If* $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}\right). \tag{2.6}$$

(2) *If there exists some* $R < \infty$ *such that*

$$\operatorname{supp} \rho_\varepsilon \subset B(0,R),\ \forall\ 0 < \varepsilon < \varepsilon_0, \tag{2.7}$$

then (2.6) *holds.*

Proposition 2.3. — *Assume* (1.1)–(1.3)*. Let* $u \in L^1_{loc}(\mathbb{R}^N)$ *be such that* $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall\ \varepsilon$.

(1) *If* $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \quad \left(\text{with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{BV}\right). \tag{2.8}$$

(2) *If* (2.7) *holds, then* (2.8) *holds.*

Open Problem 2.4. — *Let* $u \in L^1_{loc}(\mathbb{R}^N)$ *be such that* $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$*. Is it true that* (2.6)*, respectively* (2.8)*, hold,* without assuming the support assumption (2.7)?

Proof of Propositions 2.2 and 2.3. — In view of Lemma 1.4(2) and Propositions 1.7 and 1.8, it suffices to prove the following. If $\ell := \liminf_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} < \infty$, then $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in \dot{BV}$ if $p = 1$. Clearly, this holds provided

$$\int_{\mathbb{R}^N} u \operatorname{div} \zeta \leqslant \ell\, \|\zeta\|_{L^q(\mathbb{R}^N)},\ \forall\ \zeta \in C_c^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right), \tag{2.9}$$

where q is the conjugate exponent of p. In turn, (2.9) holds provided

$$-\int_{\mathbb{R}^N} u \operatorname{div} \zeta = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta, \ \forall\ \zeta \in C_c^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right). \tag{2.10}$$

In order to complete the proof, it suffices to establish (2.10) under the assumptions of Proposition 2.2, respectively 2.3 (with no boundedness assumption on $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}$).

In view of (2.4), in order to obtain (2.10) it suffices to prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} u\, V_{\varepsilon,\zeta_j} \cdot e_j = \int_{\mathbb{R}^N} u\, \partial_j \zeta_j,\ 1 \leqslant j \leqslant N. \tag{2.11}$$

When $u \in L^\infty(\mathbb{R}^N)$, this follows from Proposition 1.7 applied to ζ_j with $p = 1$.

When $u \in L^1(\mathbb{R}^N)$, we note the domination

$$\left|u\, V_{\varepsilon,\zeta_j} \cdot e_j\right| \leqslant \|\nabla \zeta_j\|_{L^\infty(\mathbb{R}^N)} |u| \in L^1\left(\mathbb{R}^N\right).$$

We conclude by dominated convergence, using the fact that $V_{\varepsilon,\zeta_j} \cdot e_j$ converges to $\partial_j \zeta_j$ pointwise as $\varepsilon \to 0$ (see Remark 1.13).

The argument for $u \in L^1_{loc}(\mathbb{R}^N)$ under the support condition (2.7) is similar.

The proof of Propositions 2.2 and 2.3 is complete. □

One may consider versions of Propositions 2.2 and 2.3 for families of functions instead of a fixed function. Here are, for example, two versions of [7, Theorem 3.7].

Proposition 2.5. — *Assume (1.1)–(1.3). Let $1 < p < \infty$. Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon,u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that*

$$(V_{\varepsilon,u_\varepsilon}) \text{ is bounded in } L^p\left(\mathbb{R}^N\right). \tag{2.12}$$

(1) *If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{W}^{1,p}$ such that, up to a subsequence $\varepsilon_k \to 0$,*

$$u_\varepsilon \rightharpoonup u \ \ast\text{-weakly in } \mathscr{M}_{loc}\left(\mathbb{R}^N\right), \tag{2.13}$$

$$\|\nabla u\|_{L^p(\Omega)} \leqslant \liminf_{\varepsilon \to 0} \|V_{\varepsilon,u_\varepsilon}\|_{L^p(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \tag{2.14}$$

(2) *If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds, then (2.13)–(2.14) hold.*

Proposition 2.6. — *Assume (1.1)–(1.3). Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon,u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that*

$$(V_{\varepsilon,u_\varepsilon}) \text{ is bounded in } L^1\left(\mathbb{R}^N\right). \tag{2.15}$$

(1) *If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{BV}$ such that, up to a subsequence $\varepsilon_k \to 0$,*

$$u_\varepsilon \rightharpoonup u \ \ast\text{-weakly in } \mathscr{M}_{loc}\left(\mathbb{R}^N\right), \tag{2.16}$$

$$\|Du\|_{\mathscr{M}(\Omega)} \leqslant \liminf_{\varepsilon \to 0} \|V_{\varepsilon,u_\varepsilon}\|_{L^1(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \tag{2.17}$$

(2) *If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds, then (2.16)–(2.17) hold.*

Remark 2.7. — Note that, by Lemma 1.3, (2.12) (respectively, (2.15)) holds if $I_{\varepsilon,p,u_\varepsilon} \leqslant C < \infty$, $\forall\ 0 < \varepsilon < \varepsilon_0$ (respectively, $I_{\varepsilon,1,u_\varepsilon} \leqslant C < \infty$, $\forall\ 0 < \varepsilon < \varepsilon_0$).

Proofs of Propositions 2.5 and 2.6. — We present the argument for Proposition 2.6; the proof of Proposition 2.5 is similar. Consider a (signed) Radon measure μ on $\mathbb{R}^N$ such that, up to a subsequence, $u_{\varepsilon_k} \rightharpoonup \mu$ $*$-weakly in $\mathscr{M}_{loc}(\mathbb{R}^N)$. Fix some $\zeta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$. We then have

$$\begin{aligned}\int_{\mathbb{R}^N} \operatorname{div}\zeta\, d\mu &= \lim_k \int_{\mathbb{R}^N} \operatorname{div}\zeta\, u_{\varepsilon_k} \\ &= \lim_k \int_{\mathbb{R}^N} \sum_j \left[V_{\varepsilon_k,\zeta_j}\cdot e_j\right]\, u_{\varepsilon_k} \\ &\quad + \lim_k \int_{\mathbb{R}^N} \left(\operatorname{div}\zeta - \sum_j \left[V_{\varepsilon_k,\zeta_j}\cdot e_j\right]\right) u_{\varepsilon_k} \\ &:= \lim_k A_k + \lim_k B_k.\end{aligned} \tag{2.18}$$

Step 1. We have $B_k \to 0$. — We have to treat three cases: (i) (u_{ε_k}) is bounded in $L^1(\mathbb{R}^N)$; (ii) (u_{ε_k}) is bounded in $L^\infty(\mathbb{R}^N)$; (iii) (u_{ε_k}) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds.

Step 1.1. Proof in case (i). — We use the fact that, by Remark 1.13, $\operatorname{div}\zeta - \sum_j[V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ uniformly in $\mathbb{R}^N$, together with the boundedness of (u_{ε_k}) in $L^1(\mathbb{R}^N)$.

Step 1.2. Proof in case (ii). — By Proposition 1.7, we have $\operatorname{div}\zeta - \sum_j[V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ in $L^1(\mathbb{R}^N)$. We combine this fact with the boundedness of (u_{ε_k}) in $L^\infty(\mathbb{R}^N)$.

Step 1.3. Proof in case (iii). — By Remark 1.13, we have $\operatorname{div}\zeta - \sum_j[V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ uniformly in $\mathbb{R}^N$. By the support condition (2.7), there exists some $r > 0$ such that, for each ε, $\operatorname{div}\zeta - \sum_j[V_{\varepsilon_k,\zeta_j}\cdot e_j] = 0$ in $\mathbb{R}^N \setminus B(0,r)$. We find that

$$|B_k| \leqslant \left\|\operatorname{div}\zeta - \sum_j \left[V_{\varepsilon_k,\zeta_j}\cdot e_j\right]\right\|_{L^\infty(\mathbb{R}^N)} \|u_{\varepsilon_k}\|_{L^1(B(0,r))} \to 0 \text{ as } k\to\infty.$$

Step 2. Conclusion. — By Lemma 2.1, we have

$$A_k = -\int_{\mathbb{R}^N} V_{\varepsilon_k,u_{\varepsilon_k}}\cdot\zeta. \tag{2.19}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Combining (2.18), Step 1, (2.19), and the assumption (2.15), we find that, when $\zeta \in C_c^\infty(\Omega;\mathbb{R}^N)$,

$$\int_\Omega \operatorname{div}\zeta\, d\mu \leqslant \|\zeta\|_{L^\infty(\Omega)} \liminf_\varepsilon \|V_{\varepsilon,u_\varepsilon}\|_{L^1(\Omega)}.$$

It follows that $\mu \in \dot{BV}$ and that (2.17) holds. □

Remark 2.8. — In view of [1, Theorem 4] and Ponce [8, Theorem 1.2], it is likely, but not known, that, in Propositions 2.5 and 2.6, the boundedness assumptions on u_ε can be removed, and that the $*$-weak convergence in $\mathscr{M}_{loc}$ can be improved to strong L^p_{loc} convergence. In this direction, we formulate below two open questions.

Open Problem 2.9. — *Let $1 \leqslant p < \infty$. Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon,u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that*

$$(V_{\varepsilon,u_\varepsilon}) \text{ is bounded in } L^p\left(\mathbb{R}^N\right), \tag{2.20}$$

$$\int_{B(0,1)} u_\varepsilon = 0, \ \forall\ \varepsilon. \tag{2.21}$$

(1) *Is it true that (u_ε) is bounded in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?*
(2) *Is it true that (u_ε) is relatively compact in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?*

Note the natural condition (2.21). Such a "normalization" condition is needed since V_ε "does not see constants"; therefore, in order to have a priori estimates, one has to "kill the constants".

3. A distributional approach

A natural generalization of the approach in the previous section (based on the identity (2.4)), consistent with the spirit of the theory of distributions, was initiated in [7]. It consists of taking the *identity* (2.4) as a *definition* of V_ε. More precisely, instead of assuming that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, we assume that (2.4) holds for every $\zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$ and some function $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *that is not, a priori, given by* (1.4). This is a *distributional version* of V_ε given by (1.4) and, by the proof of (2.4), it coincides with V_ε provided that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. One could even go one step beyond and define the *distribution* V_ε through the formula $V_\varepsilon(\zeta)$=the right-hand side of (2.4). (See also, for similar approaches in different but related settings, Shieh and Spector [9], Comi and Stefani [4], Bruè, Calzi, Comi, and Stefani [3].)

Repeating the end of the proof of the Propositions 2.2 and 2.3, we obtain, e.g., the following

Proposition 3.1. — *Assume* (1.1)–(1.3). *Let $1 < p < \infty$. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that* (2.4) *holds, $\forall\ \varepsilon$, $\forall\ \zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$, then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}$$
$$\left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}\right). \tag{3.1}$$

And the usual variants for $p = 1$ or $u \in L^1_{loc}(\mathbb{R}^N)$, under the support condition (2.7).

4. Heavy tails kernels

The results in this section are in the spirit of [4].

Let ρ_ε satisfy (1.1) and the following variants of (1.2)–(1.3):

$$\lim_{\varepsilon \to 0} \int_{|h|<1} \rho_\varepsilon(h)\, dh = 1, \tag{4.1}$$

$$\lim_{\varepsilon \to 0} \int_{|h|>\delta} \frac{\rho_\varepsilon(h)}{|h|}\, dh = 0, \ \forall\ \delta > 0. \tag{4.2}$$

Note that these assumptions are weaker than (1.2)–(1.3) and that they allow *heavy tails* kernels, which are not integrable at infinity. Here is a special case, considered, e.g., in [4], of kernels satisfying (4.1)–(4.2):

$$\rho_\varepsilon(h) := \frac{2^{1-\varepsilon}}{N\,\pi^{N/2}}\frac{\Gamma\left(\dfrac{N-\varepsilon}{2}+1\right)}{\Gamma\left(\dfrac{\varepsilon}{2}\right)}\frac{1}{|h|^{N-\varepsilon}},\ 0<\varepsilon<N+2. \tag{4.3}$$

Indeed, the validity of (4.2) is straightforward, while (4.1) follows from the fact that

$$\frac{2^{1-\varepsilon}}{N\,\pi^{N/2}}\frac{\Gamma\left(\dfrac{N-\varepsilon}{2}+1\right)}{\Gamma\left(\dfrac{\varepsilon}{2}\right)} \sim_{\varepsilon\to 0} \frac{\varepsilon}{N\,\pi^{N/2}}\Gamma\left(\frac{N}{2}+1\right)=\frac{\varepsilon}{|\mathbb{S}^{N-1}|},$$

combined with the identity

$$\frac{\varepsilon}{|\mathbb{S}^{N-1}|}\int_{|h|<1}\frac{1}{|h|^{N-\varepsilon}}\,dh=1.$$

The results in the previous sections can be easily adapted to kernels satisfying (4.1)–(4.2). The price to pay is that the natural function setting is $W^{1,p}(\mathbb{R}^N)$, respectively $BV(\mathbb{R}^N)$, rather than $\dot W^{1,p}$, respectively $\dot{BV}$. Here are some results from the previous sections adapted to the assumptions (1.1) and (4.1)–(4.2).

Proposition 4.1. — *Assume* (1.1) *and* (4.1)–(4.2). *Let* $1\leqslant p<\infty$. *Let* $u\in W^{1,p}(\mathbb{R}^N)$. *Then*

$$V_\varepsilon\to\nabla u \text{ in } L^p\left(\mathbb{R}^N\right) \text{ as } \varepsilon\to 0.$$

Proposition 4.2. — *Assume* (1.1) *and* (4.1)–(4.2). *Let* $u\in BV(\mathbb{R}^N)$. *Then*

$$V_\varepsilon\rightharpoonup Du \ *\text{-weakly in } \mathscr{M}\left(\mathbb{R}^N\right) \text{ as } \varepsilon\to 0 \tag{4.4}$$

and

$$\lim_{\varepsilon\to 0}\|V_\varepsilon\|_{L^1(\mathbb{R}^N)}=\|Du\|_{\mathscr{M}(\mathbb{R}^N)}. \tag{4.5}$$

In the next two results, we assume that, for some $1<q\leqslant\infty$, we have

$$\lim_{\varepsilon\to 0}\|\rho_\varepsilon(h)/|h|\|_{L^q(|h|>1)}=0. \tag{4.6}$$

Proposition 4.3. — *Assume* (1.1) *and* (4.1)–(4.2). *Let* $1\leqslant p<\infty$. *Assume that* (4.6) *holds when* q *is the conjugate exponent of* p. *Let* $u\in W^{1,p}(\mathbb{R}^N)$. *Then, for a.e.* $x\in\mathbb{R}^N$, *we have* $V_\varepsilon(x)\to\nabla u(x)$ *as* $\varepsilon\to 0$.

Proposition 4.4. — *Assume* (1.1) *and* (4.1)–(4.2). *Assume that* (4.6) *holds when* $q=\infty$. *Let* $u\in BV(\mathbb{R}^N)$. *Then, for a.e.* $x\in\mathbb{R}^N$, *we have* $V_\varepsilon(x)\to\nabla^{ac}u(x)$ *as* $\varepsilon\to 0$.

Remark 4.5. — Given any $q>1$, the kernel in (4.3) satisfies (4.6). Therefore, Propositions 4.3 and 4.4 apply to these kernels.

Remark 4.6. — Propositions 4.3 and 4.4 have straightforward versions, in which the assumption on u is $u\in L^r(\mathbb{R}^N)\cap\dot W^{1,p}$, respectively $u\in L^r(\mathbb{R}^N)\cap\dot{BV}$ for some $r\in[1,\infty)$ (and then, in (4.6), q is the conjugate exponent of r).

Proposition 4.7. — *Assume* (1.1) *and* (4.1)–(4.2). *Let* $1 < p < \infty$. *If* $u \in L^p(\mathbb{R}^N)$ *and* $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *for every* ε, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \\ \Big(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}\left(\mathbb{R}^N\right)\Big). \quad (4.7)$$

Proposition 4.8. — *Assume* (1.1) *and* (4.1)–(4.2). *If* $u \in L^1(\mathbb{R}^N)$ *and* $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *for every* ε, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \\ \Big(\text{with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV\left(\mathbb{R}^N\right)\Big). \quad (4.8)$$

Proposition 4.9. — *Assume* (1.1) *and* (4.1)–(4.2). *Let* $1 < p < \infty$. *If* $u \in L^p(\mathbb{R}^N)$ *and there exists some* $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *such that* (2.4) *holds*, $\forall\ \varepsilon$, $\forall\ \zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \\ \Big(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}\left(\mathbb{R}^N\right)\Big). \quad (4.9)$$

Proposition 4.10. — *Assume* (1.1) *and* (4.1)–(4.2). *If* $u \in L^1(\mathbb{R}^N)$ *and there exists some* $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *such that* (2.4) *holds*, $\forall\ \varepsilon$, $\forall\ \zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$, *then*

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \\ \Big(\text{with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)\Big). \quad (4.10)$$

We prove only Propositions 4.1 and 4.3; the other results are obtained from the corresponding ones in the previous sections using similar arguments.

Proof of Proposition 4.1. — Set

$$\rho^1_\varepsilon := \rho_\varepsilon\, \chi_{B(0,1)},\ \rho^2_\varepsilon := \rho_\varepsilon - \rho^1_\varepsilon,\ V^1_\varepsilon := V_{\varepsilon,u,\rho^1_\varepsilon},\ V^2_\varepsilon := V_{\varepsilon,u,\rho^2_\varepsilon},\ \kappa_\varepsilon(h) := \frac{\rho^2_\varepsilon(h)}{|h|}.$$

By Proposition 1.7 and the assumptions (1.1) and (4.1)–(4.2), we have $V^1_\varepsilon \to \nabla u$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$. On the other hand, we have the straightforward inequality

$$\left|V^2_\varepsilon(x)\right| \leqslant |u| * \kappa_\varepsilon(x) + |u(x)|\, \|\kappa_\varepsilon\|_{L^1(\mathbb{R}^N)},\ \forall\ x \in \mathbb{R}^N. \quad (4.11)$$

Combining (4.2), (4.11) with $\delta = 1$, and the Young inequality, we find that $V^2_\varepsilon \to 0$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$. □

Proof of Proposition 4.3. — It suffices to note that (by (4.11) and (4.6)) we have $V^2_\varepsilon \to 0$ pointwise as $\varepsilon \to 0$. □

Remark 4.11. — The fact that ρ is radial when $|h| > 1$ is not relevant for the above results.

5. Γ-convergence

One can associate Γ-convergence results with the above convergence statements. We present one result of this type, in the spirit of [7, Theorem 1.7].

Let $1 \leqslant p < \infty$. Set, for $0 < \varepsilon < \varepsilon_0$ and $u \in L^1_{loc}(\mathbb{R}^N)$,

$$J_{\varepsilon,p}(u) := \begin{cases} \|V_{\varepsilon,u}\|_{L^p(\mathbb{R}^N)}, & \text{if } W_{\varepsilon,u} \in L^1_{loc}\left(\mathbb{R}^N\right) \\ \infty, & \text{otherwise} \end{cases},$$

$$J_{0,p}(u) := \begin{cases} \|\nabla u\|_{L^p(\mathbb{R}^N)}, & \text{if } u \in \dot{W}^{1,p} \\ \infty, & \text{otherwise} \end{cases}, \ \forall\ 1 < p < \infty,$$

$$J_{0,1}(u) := \begin{cases} \|Du\|_{\mathscr{M}(\mathbb{R}^N)}, & \text{if } u \in \dot{BV} \\ \infty, & \text{otherwise} \end{cases}.$$

Proposition 5.1. — *Assume* (1.1)–(1.3). *Let* $1 \leqslant p < \infty$.

(1) *For* $1 \leqslant q < \infty$, $J_{\varepsilon,p}$ *Γ-converges to* $J_{0,p}$ *in* $L^q(\mathbb{R}^N)$.
(2) *Under the support assumption* (2.7), $J_{\varepsilon,p}$ *Γ-converges to* $J_{0,p}$ *in* $L^1_{loc}(\mathbb{R}^N)$.

Proof of item (2). — Let $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subset L^1_{loc}(\mathbb{R}^N)$ be a family such that $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^N)$ and $\liminf_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon) < \infty$. By Proposition 2.5 and the proof of Propositions 2.2 and 2.3 (see, more specifically, (2.9) and (2.10)), we find that $u \in \dot{W}^{1,p}$ if $1 < p < \infty$ (respectively $u \in \dot{BV}$ if $p = 1$) and $J_{0,p}(u) \leqslant \liminf_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon)$.

In the opposite direction, we do not need the support assumption (2.7). Let $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in \dot{BV}$ if $p = 1$. Let η be a normalized bump function. By Proposition 1.7 applied to $u * \eta_{1/j}$, where $j \geqslant 1$ is an integer, there exists a sequence $(\varepsilon_j)_{j\geqslant 1}$ such that

$$J_{\varepsilon,p}\left(u * \eta_{1/j}\right) \leqslant J_{0,p}\left(u * \eta_{1/j}\right) + \frac{1}{j},\ \forall\ j \geqslant 1,\ \forall\ 0 < \varepsilon < \varepsilon_j.$$

With no loss of generality, we may assume that $\varepsilon_j \to 0$ and $\varepsilon_{j+1} < \varepsilon_j$. If we set

$$u_\varepsilon := u * \eta_{1/j},\ \forall\ \varepsilon_{j+1} \leqslant \varepsilon < \varepsilon_j,$$

then $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^N)$ when $\varepsilon \to 0$ and

$$\limsup_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon) \leqslant \limsup_{j\to\infty} J_{0,p}(u * \eta_{1/j}) = J_{0,p}(u).$$

Proof of item (1). — The proof is similar to the one of item (2). It suffices to note that Propositions 2.2, 2.3, and 2.5 still hold if we replace $(L^1 + L^\infty)(\mathbb{R}^N)$ with $L^q(\mathbb{R}^N)$. □

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