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THE PERMUTON LIMIT OF RANDOM RECURSIVE SEPARABLE PERMUTATIONS

VALENTIN FÉRAY AND KELVIN RIVERA-LOPEZ

Abstract. We introduce and study a simple Markovian model of random separable permutations. Our first main result is the almost sure convergence of these permutations towards a random limiting object in the sense of permutons, which we call the *recursive separable permuton*. We then prove several results on this new limiting object: a characterization of its distribution via a fixed-point equation, a combinatorial formula for its expected pattern densities, an explicit integral formula for its intensity measure, and lastly, we prove that its distribution is absolutely singular with respect to that of the Brownian separable permuton, which is the large size limit of *uniform* random separable permutations.

1. INTRODUCTION

1.1. Our model. Fix $p \in (0, 1)$. We consider a sequence of random permutations $(\sigma^{(n),p})_{n \geq 1}$ starting from the unique permutation of size 1 and defined recursively. Given $\sigma^{(n),p}$, a permutation of size n , we obtain $\sigma^{(n+1),p}$, a permutation of size $n + 1$, by the following procedure.

- (1) Take j uniformly at random between 1 and n .
- (2) In the one line notation of $\sigma^{(n),p}$, we increase all values bigger than j by 1.
- (3) With probability p (resp. $1 - p$), we replace j by $j, j+1$ (resp. by $j+1, j$).

An example is given on Figure 1.1; here and throughout the paper, permutations are represented by their diagrams (the diagram of a permutation π of size n is the set of dots $(j, \pi(j))$ drawn in an $n \times n$ grid). This operation of replacing a point in the diagram by two consecutive points (consecutive at the same time in value and in position) will be referred to as *inflation*. We call the inflation *increasing* or *decreasing*, depending on the relative position of the two new points.

The set of permutations which can be obtained from the permutation 1 by repeated inflations (either increasing or decreasing) is known as the set of *separable permutations*. Alternatively, separable permutations are those permutations that avoid the patterns 3142 and 2413, see, e.g., [12]. By construction, the permutations $(\sigma^{(n),p})_{n \geq 1}$ are separable and any given separable permutation will appear in this sequence with nonzero probability. We therefore refer to $\sigma^{(n),p}$ as the *random recursive separable permutation (of size n and parameter p)*. This model differs from the model of uniform random separable permutations (studied, e.g. in [5, 22]) and, as we will see in Proposition 1.3, it yields a different object in the limit.

Remark 1.1. — It was asked in a recent survey on permutons whether uniform separable permutations can be sampled in a Markovian way [17, Section 5.4]. Though we do not answer this question here, this served as an additional motivation to study a natural Markovian model of random separable permutations.

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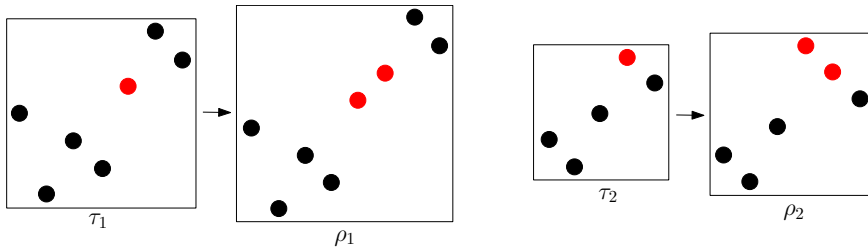


FIGURE 1.1. Examples of possible inflation steps from $\tau := \sigma^{(n),p}$ to $\rho := \sigma^{(n+1),p}$. For $i = 1$ or 2 , the point of τ_i chosen uniformly at random, as well as the two new adjacent points in ρ_i replacing it, are painted in red. In ρ_1 , these two new points are in increasing order (we say that we have performed an increasing inflation), while, in ρ_2 , they are in decreasing order (in this case, we have performed a decreasing inflation).

1.2. The limiting permuton. Throughout the paper, let Leb be the Lebesgue measure on $[0, 1]$. Recall that if ν is a measure on A and g is a measurable map from A to B , then the formula $g\#\nu(C) = \nu(g^{-1}(C))$ for any measurable subset C of B defines a measure $g\#\nu$ on B , called the *push-forward measure*. Finally, we denote by π_1 and π_2 the projection map from $[0, 1]^2$ to $[0, 1]$ on the first and second coordinates, respectively.

By definition, a permuton is a probability measure μ on the unit square $[0, 1]^2$ whose projections on the horizontal and vertical axes are both uniform, i.e. $(\pi_1)\#\mu = (\pi_2)\#\mu = \text{Leb}$. Permutons are natural limit objects for permutations of large sizes, see [17] for a recent survey on the topic. Indeed, we can encode a permutation π of size n by a permuton $\mu_\pi = \frac{1}{n} \sum_{i=1}^n \lambda(i, \pi(i))$, where $\lambda(i, \pi(i))$ is the measure of mass 1 uniformly spanned on the square $[\frac{i-1}{n}; \frac{i}{n}] \times [\frac{\pi(i)-1}{n}; \frac{\pi(i)}{n}]$. Equivalently, μ_π has a piecewise constant density

$$g(x, y) = \begin{cases} n & \text{if } \pi(\lceil nx \rceil) = \lceil ny \rceil; \\ 0 & \text{otherwise.} \end{cases}$$

A sequence of permutations $\pi^{(n)}$ then converges to a permuton μ if the associated measures $\mu_{\pi^{(n)}}$ converge to μ , in the sense of the weak convergence of measures. We can now state the first main result of this paper.

THEOREM 1.2. — *The random permutations $(\sigma^{(n),p})_{n \geq 1}$ converge a.s. to a random permuton. We call this permuton the recursive separable permuton (of parameter p) and denote it by μ_p^{rec} .*

A sample of $\sigma^{(n),p}$ for $n \in \{10, 100, 1000\}$ and $p = 1/2$ is given in Figure 1.2. In this simulation, the three permutations are taken from the *same realization* of the random process $(\sigma^{(n),p})_{n \geq 1}$. The a.s. convergence towards a limiting permuton is visible on this simulation.

This theorem should be compared with the limit result for *uniform* random separable permutations obtained in [5]. In the latter case, the limit is the so-called *Brownian separable permuton*. This Brownian separable permuton belongs to the

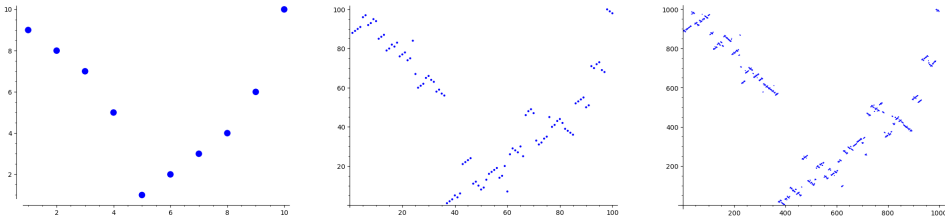


FIGURE 1.2. A sample of permutations $\sigma^{(10),0.5}$, $\sigma^{(100),0.5}$, $\sigma^{(1000),0.5}$ corresponding to the same realization of the process $(\sigma^{(n),0.5})_{n \geq 1}$.

larger family of biased Brownian separable permutons, also indexed by a parameter p in $(0, 1)$, and denoted μ_p^{Br} (we will often drop the word *biased* in the latter).

The constructions of the recursive and Brownian separable permutons share some similarities, but these are different distributions on the set of permutons. In fact, we prove in Section 4.4 the following stronger statement.

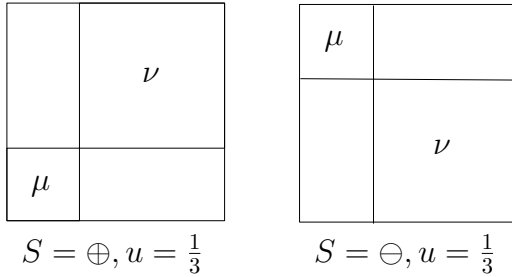
PROPOSITION 1.3. — *Let $p \neq q$ be fixed in $(0, 1)$. Then the distributions of the four permutons μ_p^{rec} , μ_q^{rec} , μ_p^{Br} and μ_q^{Br} are pairwise singular¹.*

Thus our model of increasing separable permutons yields a much different limiting object than the uniform separable permutons. This is in contrast with the result of [6], where the Brownian separable permuton was shown to be the limit of uniform random permutons in many permutation classes. It seems that going from the uniform model to a recursive model as the one studied here allows to escape the universality class of the Brownian separable permuton (see Remark 1.4 for further discussion).

Another difference with the setting of uniform permutons is that the convergence in Theorem 1.2 holds in the almost sure sense. In particular, we cannot use the criterium stating that convergence of permutons in distribution is equivalent to the convergence of expected pattern densities. Instead, we need to construct the limiting permuton on the same probability space as the process $(\sigma^{(n),p})_{n \geq 1}$ of random permutons and to prove the convergence with adhoc arguments.

Remark 1.4. — There are other cases in the literature of combinatorial objects for which the uniform measure and a natural recursive random construction yield different asymptotic behaviors. Most famously, for many families of trees, the typical height of a random tree is of order \sqrt{n} in the uniform model [2] and $\log(n)$ in recursive models [23]. A situation closer to the one of this paper in which we have nontrivial but different limits for the uniform and recursive models is that of noncrossing sets of chords in a regular n -gon. Indeed, in [14], Curien and LeGall consider recursive models of noncrossing sets of chords and show that they converge to a random limiting object L_∞ . This random set L_∞ is different from the limit of uniform random triangulations of the n -gon, previously identified by Aldous and known as the Brownian triangulation [3]. Let us also mention recent results on random graphs with fixed degree sequences and random chirotopes (chirotopes are

¹We recall that two measures μ and ν are singular if there exists a measurable set A such that $\mu(A) = 1$ but $\nu(A) = 0$.

FIGURE 1.3. The permuton $\mu \otimes_{(u,S)} \nu$.

combinatorial objects encoding the relative positions of a set of points in the plane), which both exhibit differences in the asymptotic behaviors between recursive and uniform models [16, 21]. We do not have a satisfactory intuitive explanation of these facts.

Remark 1.5. — In the theory of permutons (see, e.g., [18, Lemma 4.2]), there is an explicit construction for permutations that converge almost surely to a given permuton. It is natural to compare the permutations obtained by applying this construction to the random permuton μ_p^{rec} with the recursive separable permutations. As sequences, these two objects are clearly different: the recursive separable permutations are constructed via repeated inflations while the other sequence is not. However, we shall see in Section 4.2 that these sequences do have the same marginal distributions.

1.3. Properties of the recursive separable permuton.

1.3.1. *Self-similarity.* The distribution of the recursive separable permuton μ_p^{rec} can be characterized by a fixed-point equation. To state this property, we first need to introduce some notation.

Given two permutons μ and ν , a real number u in $[0, 1]$ and a sign S in $\{\oplus, \ominus\}$, we construct a new permuton $\rho = \mu \otimes_{(u,S)} \nu$ as follows. If $S = \oplus$, we let ρ be supported on $[0, u]^2 \cup [u, 1]^2$ such that the restriction $\rho|_{[0,u]^2}$ (resp. $\rho|_{[u,1]^2}$) is a rescaled version of μ of total weight u (resp. a rescaled version of ν of total weight $1 - u$). If $S = \ominus$, then ρ is defined similarly, but is supported on $[0, u] \times [1 - u, 1] \cup [u, 1] \times [0, 1 - u]$. We refer to Figure 1.3 for an illustration.

Now, given a random permuton μ , we denote by $\Phi_p(\mu)$ the random permuton $\mu_0 \otimes_{(U,S)} \mu_1$, where μ_0 and μ_1 are two copies of μ , U is a uniform r.v. in $[0, 1]$ and S is a random sign in $\{\oplus, \ominus\}$ with $\mathbb{P}(S = \oplus) = p$, all variables μ_0 , μ_1 , U and S being independent.

PROPOSITION 1.6. — For any p in $[0, 1]$, we have $\mu_p^{\text{rec}} \stackrel{d}{=} \Phi_p(\mu_p^{\text{rec}})$. Moreover, the distribution of μ_p^{rec} is characterized by this property, in the following sense: if a random permuton ν satisfy $\nu \stackrel{d}{=} \Phi_p(\nu)$, then $\nu \stackrel{d}{=} \mu_p^{\text{rec}}$.

1.3.2. *Expected pattern densities.* We recall that if π is a pattern (i.e. a permutation) of size k and μ a permuton, we can define the random permutation $\text{Sample}(\mu; k)$ and the pattern density $\text{dens}(\pi, \mu)$ of π in μ as follows. Let (x_i, y_i) be i.i.d. points in $[0, 1]^2$ with distribution μ . We reorder them as $(x_{(1)}, y_{(1)}), \dots, (x_{(k)}, y_{(k)})$ such

that $y_{(1)} < \dots < y_{(k)}$. Then there exists a unique (random) permutation τ such that $x_{(\tau_1)} < \dots < x_{(\tau_k)}$. This random permutation τ is denoted $\text{Sample}(\mu; k)$. We also write

$$\text{dens}(\pi, \mu) = \mathbb{P}(\text{Sample}(\mu; k) = \pi).$$

These functionals play a key role in the theory of permutons. In particular, convergence of permutons is equivalent to convergence of all pattern densities, see [18]. Also the distribution of a random permuton is uniquely determined by its expected pattern densities [6, Proposition 2.4]. The next proposition provides combinatorial descriptions of these expected pattern densities in the case of the recursive separable permuton.

To state it, we introduce some terminology. Let π and σ be two permutations of respective sizes k and ℓ . Their *direct sum* and *skew sum* are the permutations of size $k + \ell$ defined in one-line notation as follows

$$\begin{aligned} \pi \oplus \sigma &= \pi_1 \dots \pi_k (\sigma_1 + k) \dots (\sigma_\ell + k); \\ \pi \ominus \sigma &= (\pi_1 + \ell) \dots (\pi_k + \ell) \sigma_1 \dots \sigma_\ell. \end{aligned}$$

Examples with $k = 3$ and $\ell = 2$ are provided in Figure 1.4. They illustrate the graphical interpretation of these operations on permutation diagrams.

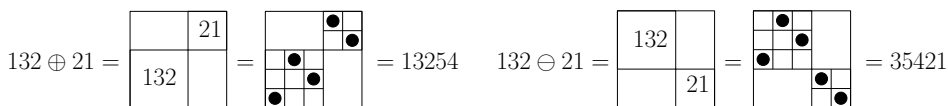


FIGURE 1.4. Direct sum and skew sum of permutations.

In the following, we consider rooted (complete) binary trees, meaning that every internal node has exactly two ordered children. Additionally, internal nodes are labeled with numbers from 1 to some k (where each integer in this range is used exactly once). The tree is said to be increasing if labels are increasing on any path from the root to a leaf. Finally, each internal node carries a decoration, which is either \oplus or \ominus . To such a tree T , we associate a permutation $\sigma = \text{Perm}(T)$ as follows (an example of a rooted increasing binary tree and the associated permutation is given in Figure 1.5).

- If T is reduced to a single leaf, then σ is the one-element permutation 1.
- Otherwise, the root of T has two (ordered) children and we call T_1 and T_2 the subtree rooted at these children. Let σ_1 and σ_2 be the permutations associated with T_1 and T_2 . Then we associate $\sigma_1 \otimes \sigma_2$ with T , where \otimes is the decoration of the root of T ($\otimes \in \{\oplus, \ominus\}$).

We note that $\text{Perm}(T)$ does not depend on the labeling of the internal nodes of T (only on the shape of T and on the decorations of its internal nodes). By construction, $\text{Perm}(T)$ is always a separable permutation. A separable permutation σ is in general associated with more than one tree T . We note that, except for the labeling of internal nodes of T , this is a standard construction in the theory of separable permutations, see, e.g., [12].

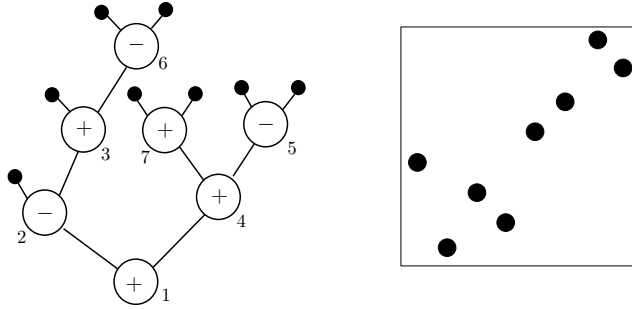


FIGURE 1.5. A rooted increasing binary tree and the associated permutation. The permutation associated to the left subtree of the root is 4132, the one associated to its right-subtree 1243. Since the root has decoration \oplus , the permutation associated to the whole tree is $4132 \oplus 1243$, which is equal to 41325687.

PROPOSITION 1.7. — For any pattern π of size n , we have

$$\mathbb{E}[\text{dens}(\pi, \boldsymbol{\mu}_p^{\text{rec}})] = \mathbb{P}[\sigma^{(n),p} = \pi] = \frac{N_{\text{inc}}(\pi)}{(n-1)!} (1-p)^{\text{des}(\pi)} p^{n-1-\text{des}(\pi)},$$

where $\text{des}(\pi)$ is the number of descents in π and $N_{\text{inc}}(\pi)$ the number of increasing binary trees T such that $\text{Perm}(T) = \pi$.

If π is not a separable pattern, then $N_{\text{inc}}(\pi) = 0$, implying $\text{dens}(\pi, \boldsymbol{\mu}_p^{\text{rec}}) = 0$ a.s.

Remark 1.8. — The fact that $\text{dens}(\pi, \boldsymbol{\mu}_p^{\text{rec}}) = 0$ a.s. for non-separable patterns π implies that the distribution of $\boldsymbol{\mu}_p^{\text{rec}}$ is also singular with respect to that of the so-called skew Brownian permutons $\mu_{\rho,q}$ with parameter (ρ, q) in $(-1, 1) \times (0, 1)$. Indeed, the latter satisfy $\text{dens}(\pi, \boldsymbol{\mu}_p^{\text{rec}}) > 0$ a.s., see [11, Theorem 1.10].

1.3.3. *The intensity measure.* Permutons are measures, so that random permutons are random measures. Given a random measure $\boldsymbol{\mu}$, one can define its *intensity measure* $I\boldsymbol{\mu}$, sometimes also denoted $\mathbb{E}\boldsymbol{\mu}$ as follows: for any measurable set A of the ground space, we have $I\boldsymbol{\mu}(A) = \mathbb{E}[\boldsymbol{\mu}(A)]$.

Our next result is a simple description of the intensity measure of the random permuton $\boldsymbol{\mu}_p^{\text{rec}}$ in terms of beta distributions. Recall that the distribution $\beta(a, b)$ with positive parameters a and b is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} dx.$$

PROPOSITION 1.9. — The intensity measure $I\boldsymbol{\mu}_p^{\text{rec}}$ of the recursive separable permuton is the distribution of

$$(U, UX_p + (1-U)X'_p),$$

where U , X_p and X'_p are independent random variables in $[0, 1]$, with distribution Leb , $\beta(p, 1-p)$ and $\beta(1-p, p)$ respectively.

From this, we can get an explicit formula for the density of $I\boldsymbol{\mu}_p^{\text{rec}}$.

COROLLARY 1.10. — $I\mu_p^{\text{rec}}$ is absolutely continuous with respect to Lebesgue measure on $[0, 1]^2$ and has density

$$\frac{1}{\Gamma(p)^2\Gamma(1-p)^2} \int_{\max(x+y-1,0)}^{\min(x,y)} \frac{dz}{z^{1-p}(x-z)^p(y-z)^p(1-x-y+z)^{1-p}}.$$

Figure 1.6 shows 3D plots of the density of $I\mu_p^{\text{rec}}$ for $p = 0.5$ and $p = 0.6$ (obtained with Mathematica). We note that the density diverges for $x = y$ when $p \geq 1/2$ and for $x + y = 1$ when $p \leq 1/2$. This is different from the Brownian separable permutation case where the density of the intensity measure diverges only in the corners (i.e. when both x and y are 0 or 1; see [20] for an explicit formula for the density in this case).

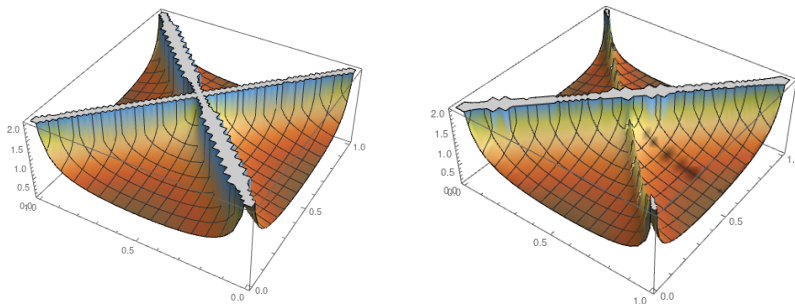


FIGURE 1.6. 3D plots of the density of $I\mu_p^{\text{rec}}$ for $p = 0.5$ (left) and $p = 0.6$ (right).

We conclude this paragraph with a discussion presenting the intensity measure of the recursive separable permutation as a limit of a natural discrete object. This interpretation is a motivation for computing the intensity measure, and is not needed later in the article. Note also that this is not specific to the recursive separable permutation; a similar discussion could be made for other models of permutations converging to random permutations.

For $n \geq 1$ and $N \geq 1$, let $\sigma_1^{(n),p}, \dots, \sigma_N^{(n),p}$ be independent copies of the recursive random separable permutation $\sigma^{(n),p}$. We then consider the average of the associated permutations:

$$\mu_N^{(n),p} = \frac{1}{N} \sum_{i=1}^N \mu_{\sigma_i^{(n),p}}. \quad (1.1)$$

This is a measure on $[0, 1]^2$ with piecewise constant density

$$g_N^{(n),p}(x, y) = \frac{n}{N} \sum_{i=1}^N \mathbb{1} \left[\sigma_i^{(n),p}(\lceil nx \rceil) = \lceil ny \rceil \right].$$

When N tends to infinity, $\mu_N^{(n),p}$ converges to $I\mu_{\sigma^{(n),p}}$, the intensity measure of the random permutation associated to $\sigma^{(n),p}$. This measure $I\mu_{\sigma^{(n),p}}$ in turn converges to $I\mu_p^{\text{rec}}$ as n tends to $+\infty$, as a consequence of Theorem 1.2. Therefore, for large n and N with $N \gg n$, the empirical average measures $\mu_N^{(n),p}$ defined in (1.1) can

be seen as a discrete approximation of $I\mu_p^{\text{rec}}$. On Figure 1.7, we plot the density $g_N^{(n),p}$ for $n = 200$, $N = 10000$ and $p \in \{.5, .6\}$. The convergence to that of $I\mu_p^{\text{rec}}$ (Fig. 1.6) is plausible on the pictures.

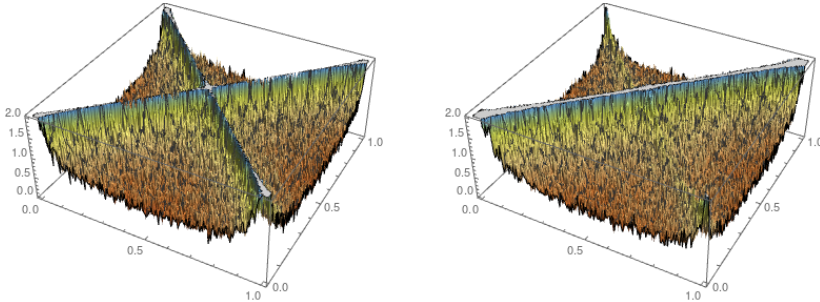


FIGURE 1.7. 3D plots of the density $g_N^{(n),p}$ for $n = 200$, $N = 10000$ and p being either 0.5 (left) or 0.6 (right).

1.4. Analog results for cographs. With a permutation σ of size n , it is standard to associate its inversion graph G_σ on vertex set $\{1, \dots, n\}$. By definition, $\{i, j\}$ is an edge of G_σ if and only if it is an inversion in σ , i.e. $(i - j)(\sigma(i) - \sigma(j)) < 0$. Inversion graphs of separable permutation are called *cographs*. Cographs can alternatively be described as graphs avoiding the path P_4 on four vertices as induced subgraph, or as graphs that can be obtained starting from single vertex graphs and iterating “disjoint union” and “taking the complement” operations. We refer to the introduction of [7] for more background on cographs.

Uniform random cographs have recently been studied in [4, 7, 24]. Considering the inversion graphs of random recursive separable permutations yields a natural Markovian model of random cographs. It can be described directly on graphs, without going through permutations.

Namely, we consider a sequence of random graphs $(G^{(n),p})_{n \geq 1}$ starting from the unique graph with one vertex and defined recursively. Given $G^{(n),p}$, a graph with n vertices, we obtain $G^{(n+1),p}$, a graph with $n+1$ vertices, by the following procedure.

- (1) Let v be a uniform random vertex of $G^{(n),p}$.
- (2) Add a new vertex v' to $G^{(n),p}$, with the same set of neighbours as v .
- (3) With probability $1 - p$, we connect v and v' with an edge.

A simulation of this random graph process for $p = 1/2$ is shown on Figure 1.8. Here graphs are represented by a collection of dots, namely two dots at coordinates (i, j) and (j, i) for each edge $\{i, j\}$ in the graph (in some sense, this is a pictorial version of the adjacency matrix of the graph). We now state a convergence result for $G^{(n),p}$, which is the analogue of Theorem 1.2. We assume the reader to be familiar with the notion of graphon convergence.

THEOREM 1.11. — *The random graphs $(G^{(n),p})_{n \geq 1}$ converge a.s. to a random graphon, which we call recursive cographon (of parameter p) and denote by $\mathbf{W}_p^{\text{rec}}$.*

A representative of the limiting graphon $\mathbf{W}_p^{\text{rec}}$ can be constructed using the random order \prec on $[0, 1]$, which we define later in Section 3. Namely, for $x < y$ in

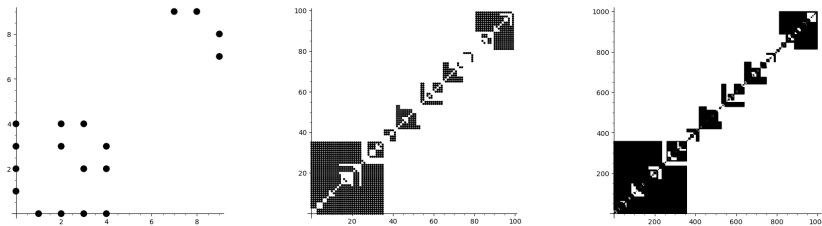


FIGURE 1.8. A sample of graphs $G^{(10), 0.5}$, $G^{(100), 0.5}$, $G^{(1000), 0.5}$ corresponding to the same realization of the process $(G^{(n), 0.5})_{n \geq 1}$.

$[0, 1]$ we set

$$\mathbf{W}_p^{\text{rec}}(x, y) = \mathbf{W}_p^{\text{rec}}(y, x) = \begin{cases} 0 & \text{if } x \prec y; \\ 1 & \text{if } y \prec x. \end{cases}$$

Moreover, $\mathbf{W}_p^{\text{rec}}$ has the following properties, which are analogues of Propositions 1.3, 1.6 and 1.7.

PROPOSITION 1.12. — *Let $p \neq q$ be fixed in $(0, 1)$. Then the distributions of the four random graphons $\mathbf{W}_p^{\text{rec}}$, $\mathbf{W}_q^{\text{rec}}$, \mathbf{W}_p^{Br} and \mathbf{W}_q^{Br} are pairwise singular, where \mathbf{W}_p^{Br} is the Brownian cographon of parameter p introduced in [7, 24].*

PROPOSITION 1.13. — *Fix p in $[0, 1]$, and let W_1 and W_2 be independent copies of $\mathbf{W}_p^{\text{rec}}$. Let also U be a uniform random variable in $[0, 1]$ and S be a Bernoulli random variable of parameter $1 - p$, independent from each other and from (W_1, W_2) . We define a graphon W by*

$$W(x, y) = \begin{cases} W_1(x/U, y/U) & \text{if } x, y \leq U; \\ W_2((x - U)/(1 - U), (y - U)/(1 - U)) & \text{if } x, y > U; \\ S & \text{if } x \leq U < y \text{ or } y \leq U < x. \end{cases}$$

Then W has the same law as $\mathbf{W}_p^{\text{rec}}$. Moreover the law of $\mathbf{W}_p^{\text{rec}}$ is characterized by this property.

For the last statement, we write $\text{dens}(H, W)$ for the (induced) density of H in a graphon W . We also recall that cographs can be encoded by decorated trees called cotrees, see, e.g., [7, Section 2.2]. Here, we will consider cotrees where internal nodes are labeled by integers from 1 to some k , and say that a cotree is increasing if labels are increasing from the root to the leaves.

PROPOSITION 1.14. — *For any graph H of size n , we have*

$$\mathbb{E}[\text{dens}(H, \mathbf{W}_p^{\text{rec}})] = \mathbb{P}[G^{(n), p} = H] = \frac{N_{\text{inc}}(H)}{(n-1)!} p^{Z(H)} (1-p)^{n-1-Z(H)},$$

where $N_{\text{inc}}(H)$ the number of increasing binary cotrees T encoding H , and $Z(H)$ is the number of decorations 0 in any binary cotree encoding H .

All of these results are easily obtained, either by applying the inversion graph mapping to the permutation results, or by adapting the proofs. Since the space of graphons has no natural convex structure, there is no natural notion of expectation of the random graphon \mathbf{W}^{rec} , and Proposition 1.9 and Corollary 1.10 have no analogues for \mathbf{W}^{rec} .

1.5. Outline of the paper. The remainder of this paper is organized as follows. In Section 2, we discuss some background material that is needed later. In Section 3, we go through an explicit construction of the recursive separable permuton and prove Theorem 1.2. In Section 4, we investigate the properties of the recursive separable permuton and prove all of our other results.

2. BACKGROUND

This section gathers some material needed in the rest of the paper. The first two items (permutation patterns and the Wasserstein metric) consist of standard material. The last item (push-forward permutons) is more specific to this project.

2.1. Permutation patterns. If σ is a permutation of size n (we simply write “permutation of n ” from now on) and I a subset of $\{1, \dots, n\}$ with k elements, then the *pattern* induced by σ on the set of positions I is the unique permutation $\tau = \text{pat}_I(\sigma)$ of k with the following property: writing $I = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$, we have, for all $1 \leq g, h \leq k$,

$$\sigma(i_g) < \sigma(i_h) \Leftrightarrow \tau(g) < \tau(h).$$

In other words, τ is obtained by considering the subsequence $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ of σ and replacing the smallest element by 1, the second smallest by 2, and so on. For example, the pattern induced by 3256471 on positions $\{2, 3, 5\}$ is 132.

2.2. Wasserstein metric. Some arguments in Sections 4.1 and 4.3 use the notion of the Wasserstein distance between probability distributions. We briefly recall some main facts about it.

Let X be a complete metric space with distance d_X . For each $p \geq 1$, consider the space $\mathcal{M}_1^p(X)$ of probability measures on X with a finite p^{th} moment. This space can be endowed with the so-called p^{th} Wasserstein distance (also called the optimal cost distance or the Kantorovich–Rubinstein distance):

$$d_{W,p}(\nu, \nu')^p := \inf_{\substack{\mathbf{X}, \mathbf{X}' : \\ \mathbf{X} \sim \nu, \mathbf{X}' \sim \nu'}} \mathbb{E} [d_X(\mathbf{X}, \mathbf{X}')^p],$$

where the infimum is taken over all pairs $(\mathbf{X}, \mathbf{X}')$ of random variables defined on the same probability space with distributions ν and ν' , respectively. It is well-known that this defines a metric on $\mathcal{M}_1^p(X)$; see, e.g., [13].

In this article, we will be interested in the cases $X = [0, 1]$ and $X = [0, 1]^2$. In these cases, and more generally whenever X is compact, we have $\mathcal{M}_1^p(X) = \mathcal{M}_1(X)$ for all p . Also, convergence in the Wasserstein metric is equivalent to the weak convergence of measures (in general, on non necessarily compact spaces, it is equivalent to weak convergence and convergence of the p^{th} moment). In particular, it is sometimes fruitful to think of permuton convergence as convergence for the (first) Wasserstein distance.

2.3. Push-forward permutons. Given a function $f : [0, 1] \rightarrow [0, 1]$, we can consider the measure $\mu_f := (\cdot, f(\cdot))_{\#} \text{Leb}$ on $[0, 1]^2$, i.e. the push-forward of the Lebesgue measure on $[0, 1]$ by the map $x \mapsto (x, f(x))$. The projection of μ_f on the first coordinate is always the Lebesgue measure on $[0, 1]$, while its projection on the second coordinate is $f_{\#} \text{Leb}$. Thus, if f preserves the Lebesgue measure, then

μ_f is a permuton. Such permutons will be referred here as *push-forward* permutons. Both the recursive and the Brownian separable permutons are push-forward permutons (for random functions f).

The permuton μ_π associated to a permutation is not a push-forward permuton. We can, however, approximate it by a push-forward measure easily. Given a permutation π of n , let us define the function $f_\pi: [0, 1] \rightarrow [0, 1]$ by $f_\pi(0) = 0$ and, for $x > 0$,

$$f_\pi(x) = \frac{\pi(\lceil nx \rceil)}{n}.$$

We also write $\widehat{\mu}_\pi = \mu_{f_\pi} = (\cdot, f_\pi(\cdot))_{\#} \text{Leb}$. The measure $\widehat{\mu}_\pi$ is not a permuton (its projection on the y -axis is not uniform), but it resembles the permuton μ_π : while μ_π has, for each i , a mass $1/n$ uniformly distributed on the square $[\frac{i-1}{n}; \frac{i}{n}] \times [\frac{\pi(i-1)}{n}; \frac{\pi(i)}{n}]$, the measure $\widehat{\mu}_\pi$ has the same mass distributed on the segment $[\frac{i-1}{n}; \frac{i}{n}] \times \{\frac{\pi(i)}{n}\}$. This clearly implies

$$d_{W,1}(\widehat{\mu}_\pi, \mu_\pi) \leq \frac{1}{n} \tag{2.1}$$

for any permutation π of n . It follows that, given a sequence of permutations $\pi^{(n)}$ of increasing size, the sequences $\{\widehat{\mu}_{\pi^{(n)}}\}_{n \geq 1}$ and $\{\mu_{\pi^{(n)}}\}_{n \geq 1}$ have the same limit points in $\mathcal{M}_1([0, 1]^2)$.

We end this section with a convergence criterium for push-forward measures μ_f , which will be used in the proof of our main result. Since particular cases of push-forward permutons have been studied in the literature [5, 9, 10], it might also be useful in other contexts. We also refer the reader to a paper of Bhattacharya and Mukherjee [8], for a related result connecting pointwise convergence of random permutations seen as functions, and convergence of the associated random permutons.

PROPOSITION 2.1. — *Let f, f_1, f_2, \dots be measurable functions on $[0, 1]$ with values in $[0, 1]$. Then the following statements are equivalent:*

- (i) $f_n \rightarrow f$ in L^p for some $p \in [1, \infty)$,
- (ii) $f_n \rightarrow f$ in L^p for all $p \in [1, \infty)$,
- (iii) $(\cdot, f_n(\cdot))_{\#} \text{Leb} \rightarrow (\cdot, f(\cdot))_{\#} \text{Leb}$ weakly.

Proof. — Letting U be a uniform random variable in $[0, 1]$ on some probability space Ω , the convergence of f_n to f in $L^p([0, 1])$ is equivalent to the convergence of the random variables $f_n(U)$ to $f(U)$ in $L^p(\Omega)$. Since all $f_n(U)$ and $f(U)$ are uniformly bounded, (i) is equivalent to (ii), and they are both equivalent to the weak convergence of $f_n(U)$ to $f(U)$.

It remains to show the equivalence with (iii). If (iii) holds, the pair $(U, f_n(U))$ converges weakly to $(U, f(U))$. Restricting to the second coordinates, we know that $f_n(U)$ converges weakly to $f(U)$, implying (i).

Conversely, if (i) holds, $f_n(U)$ converges weakly to $f(U)$, and thus $(U, f_n(U))$ converges weakly to $(U, f(U))$, i.e. (iii) holds. This completes the proof of the proposition. \square

2.4. Permutations and pairs of total orders. In some constructions, it will be convenient to see a permutation as a finite set endowed with two total orders. This is a standard point of view in the context of permutation patterns, see, e.g., [1].

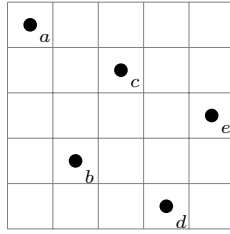


FIGURE 2.1. A set of points $E = \{a, b, c, d, e\}$; comparing x -coordinates and y -coordinates yield two orders $<$ and \prec on E . The associated permutation $\sigma = \text{Perm}(E, <, \prec)$ is 52413.

Let us explain how to associate a permutation with a triple $(E, <, \prec)$, where E is a finite set equipped with two total orders $<$ and \prec . Indeed, using the first order, we write $E = \{x_1, \dots, x_k\}$ where $k = |E|$ and $x_1 < \dots < x_k$. Then there exists a unique permutation σ satisfying

$$\sigma(j) < \sigma(k) \iff x_j \prec x_k. \quad (2.2)$$

We denote this permutation by $\sigma = \text{Perm}(E, <, \prec)$. For example, we choose $E = \{a, b, c, d, e\}$, with $a < b < c < d < e$ and $d \prec b \prec e \prec c \prec a$, then $\sigma = 52413$. To visualize this construction, we can represent elements of E as points in the plane, so that $<$ compares the x -coordinates, while \prec compares the y -coordinates. Then E resembles the diagram of σ , see Figure 2.1.

Taking patterns is simple with this viewpoint. If $E = \{x_1, \dots, x_k\}$ is as above (in particular, assuming $x_1 < \dots < x_k$) and if I is a subset of $\{1, \dots, k\}$, then

$$\text{pat}_I(\text{Perm}(E, <, \prec)) = \text{Perm}(\{x_i, i \in I\}, <, \prec).$$

Finally, $\sigma = \text{Perm}(E, <, \prec)$ is also given by an explicit formula. As above, let x_k be the k^{th} smallest element in E for the first order $<$. Then, if x_k is the ℓ^{th} smallest element for the order \prec , we have

$$\sigma(k) = \ell = 1 + \sum_{x \in E} \mathbf{1}[x \prec x_k]. \quad (2.3)$$

3. CONSTRUCTION AND CONVERGENCE

In this section, we construct the recursive separable permutations and the recursive separable permutation on a common probability space and establish the almost sure convergence of the permutations to the permutation.

Fix p in $(0, 1)$. Throughout the section, we consider two independent random i.i.d. sequences $(U_j)_{j \geq 1}$ and $(S_j)_{j \geq 1}$, where the U_j are uniform in $[0, 1]$ and the S_j are random signs in $\{\oplus, \ominus\}$ with $\mathbb{P}(S_j = \oplus) = p$. For convenience, the signs \oplus and \ominus will often be regarded as 1 and -1 , respectively, and we set $U_0 = 0$ and $U_{-1} = 1$.

To illustrate our construction, we sampled sequences $(U_j)_{j \geq 1}$ and $(S_j)_{j \geq 1}$ as above (setting $p = 1/2$), and used them in examples throughout the section. The first sampled values (rounded to 2 decimal digits for U_j) are

$$\begin{aligned} U_1 &= 0.72, & U_2 &= 0.82, & U_3 &= 0.54, & U_4 &= 0.13, & U_5 &= 0.60, \\ S_1 &= \oplus, & S_2 &= \ominus, & S_3 &= \ominus, & S_4 &= \ominus, & S_5 &= \oplus. \end{aligned} \quad (3.1)$$

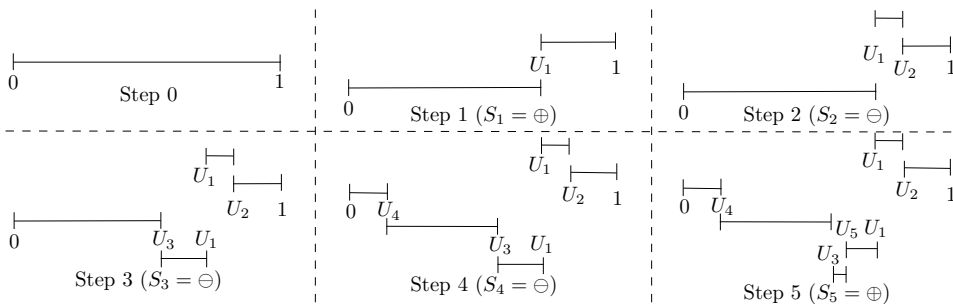


FIGURE 3.1. Illustration of the first five partial orders associated with the sequences $(U_j)_{j \leq 5}$ and $(S_j)_{j \leq 5}$ given in (3.1). An interval drawn at a higher level than another one means that its elements are larger.

3.1. The general strategy. To construct our permutations and permutation, we will construct a sequence of partial orders on $[0, 1]$ that will give rise to a total order on $[0, 1]$ in “the limit”. These partial orders will then be identified as permutations (in a similar spirit as in Section 2.4), while the total order will lead to a permutation.

With $(U_j)_{j \geq 1}$ and $(S_j)_{j \geq 1}$ defined as above, we define our first partial order by using U_1 to split $[0, 1]$ into the two intervals $[0, U_1)$ and $[U_1, 1]$ and declaring that every element in $[0, U_1)$ is less than every element in $[U_1, 1]$ if $S_1 = \oplus$ and that $[U_1, 1]$ is instead the smaller interval if $S_1 = \ominus$. The second partial order is to be obtained from the first one by using U_2 to further split one of the intervals into two new intervals and declare that the leftmost interval is the smaller of the two if $S_2 = \oplus$ and that it is the larger of the two if instead $S_2 = \ominus$. We repeat this process to obtain a sequence of partial orders, at each step using the next uniform random variable to split an interval into two intervals and declaring the leftmost interval to be the smaller interval when the corresponding sign is \oplus and the larger interval otherwise. See Figure 3.1.

We show that these partial orders can be identified as the recursive separable permutations in Section 3.5, but this can be seen intuitively. Indeed, the splitting of intervals into two and then assigning a sign to compare them mimics the inflation operation. Moreover, the distribution of the signs ensures that increasing adjacencies are created at the right rate. Finally, the properties of the uniform sequence ensure that the interval to be split at each step is chosen uniformly at random, as in our permutation model. Another motivation for the choice of the uniform sequence is the fact that the proportion of intervals to the left of U_1 must resemble the proportion of points in $\sigma^{(n),p}$ that were obtained from subdividing the leftmost point in $\sigma^{(2),p}$. The number of such points has the same law as the number of white balls in the standard Pólya urn process, and its proportion is known to converge a.s. to a uniform random variable in $[0, 1]$.

The above partial orders naturally lead in the limit $n \rightarrow \infty$ to a total order on $[0, 1]$. Thus we expect that the limiting permutation can be retrieved from this order. This retrieval is executed in Section 3.2, as a natural extension of the construction of Section 2.4.

3.2. The total order and the limiting permuton. We start by offering an explicit formal description of the order described informally in the previous section. Given $x < y$ in $[0, 1]$ such that $(x, y] \cap \{U_j\}_{j \geq 1} \neq \emptyset$, we consider the minimal $j \geq 1$ such that $U_j \in (x, y]$. We set

$$\begin{cases} x \prec y & \text{if } S_j = \oplus; \\ y \prec x & \text{if } S_j = \ominus. \end{cases} \quad (3.2)$$

Alternatively, letting

$$\begin{aligned} \mathcal{I}_{x,y} &= \left(\min(x, y), \max(x, y) \right], & x, y \in [0, 1], \\ i_{x,y} &= \inf \{j \geq 1 : U_j \in \mathcal{I}_{x,y}\}, & x, y \in [0, 1], \end{aligned}$$

and using the convention

$$S_\infty = 0,$$

we can describe this relation concisely as

$$x \prec y \iff (y - x)S_{i_{x,y}} > 0.$$

Using the random relation \prec , we define a random function $\phi: [0, 1] \rightarrow [0, 1]$ and a random measure on $[0, 1]^2$ by

$$\phi(x) = \text{Leb}(\{y \in [0, 1] : y \prec x\}), \quad (3.3)$$

$$\mu_p^{\text{rec}} = \mu_\phi = (\cdot, \phi(\cdot))_{\#} \text{Leb}, \quad (3.4)$$

where, in the last equation, we use the notation μ_f of Section 2. We note that (3.3) is a natural infinite version of (2.3), while (3.4) mimicks the way we associate a measure $\widehat{\mu}_\pi$ with a permutation π (see Section 2.3).

In Proposition 3.7 below, we will see that the following properties hold almost surely: the relation \prec is a total order on $[0, 1]$, the function ϕ is Lebesgue-preserving, and the measure μ_p^{rec} is a permuton. We define the *recursive separable permuton (of parameter p)* as the random measure μ_p^{rec} on $[0, 1]^2$.

Remark 3.1. — Our construction is inspired by the construction of the (biased) Brownian separable permuton μ_p^{Br} given in [20]. Indeed, the construction of μ_p^{Br} also goes through the construction of a random order \prec . However, instead of taking i.i.d. uniform random variables $\{U_j\}_{j \geq 1}$, we consider a Brownian excursion ϵ on $[0, 1]$ and attach signs to its local minima. We then define, for $x < y$ in $[0, 1]$,

$$\begin{cases} x \prec y & \text{if } S_m = \oplus; \\ y \prec x & \text{if } S_m = \ominus, \end{cases}$$

where S_m is the sign attached to the local minimum m of ϵ on the interval $[x, y]$. The rest of the construction, namely Eqs. (3.3) and (3.4), is the same in the constructions of μ_p^{Br} and μ_p^{rec} . As said in the introduction, even though the constructions are similar, the resulting random permutons are different, and even more, their distributions are singular to each other (see Section 4.4).

3.3. The Glivenko–Cantelli theorem and some consequences. For μ_p^{rec} to be a permuton, we need the sequence $(U_j)_{j \geq 1}$ to satisfy some good properties. The fact that these properties hold a.s. is essentially given by the well-known Glivenko–Cantelli theorem, which we now recall, together with straightforward consequences. To this end, we consider the following objects, for any $n \geq 1$:

- the ordered statistics of $\{U_{-1}, U_0, \dots, U_n\}$,

$$0 = U_{(0,n)} \leq U_{(1,n)} \leq \dots \leq U_{(n,n)} \leq U_{(n+1,n)} = 1;$$

- the length of the largest interval in $[0, 1] \setminus \{U_1, \dots, U_n\}$,

$$\Delta_n = \max_{1 \leq j \leq n+1} U_{(j,n)} - U_{(j-1,n)},$$

and

- the empirical measures

$$P_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{U_j},$$

where δ_x is the Dirac measure at x .

In addition, we consider the event

$$\mathcal{E} = \left\{ (U_j)_{j \geq 1} \text{ are distinct, } \sup_{x \in [0,1]} \left| x - \frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq x) \right| \xrightarrow{n \rightarrow \infty} 0 \right\}. \quad (3.5)$$

The following result summarizes the properties of these objects that will be useful to us.

PROPOSITION 3.2. — *The event \mathcal{E} occurs almost surely and on \mathcal{E} , the following statements hold (all limits are to be taken as $n \rightarrow \infty$):*

- (i) $\sup_{t \in [0,1]} |U_{(\lceil nt \rceil, n-1)} - t| \rightarrow 0$,
- (ii) $\Delta_n \rightarrow 0$,
- (iii) for every $k \geq 1$, $\sup_{J \in \mathcal{A}_k} |\text{Leb}(J) - P_n(J)| \rightarrow 0$, where \mathcal{A}_k consists of all subsets of $[0, 1]$ that are a disjoint union of at most k intervals.

Proof. — The fact that \mathcal{E} occurs almost surely is a classical result known as the Glivenko–Cantelli Theorem. We assume for the remainder of the proof that \mathcal{E} occurs.

Since $U_{(\lceil nt \rceil, n-1)} \in [0, 1]$ for all $t \in [0, 1]$ and $n \geq 1$, we can apply the limit in (3.5): as $n \rightarrow \infty$, we have the convergence

$$\sup_{t \in [0,1]} \left| U_{(\lceil nt \rceil, n-1)} - \frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq U_{(\lceil nt \rceil, n-1)}) \right| \rightarrow 0. \quad (3.6)$$

Recalling that the sequence $(U_j)_{j \geq 1}$ contains distinct values, we obtain the estimate

$$\left| \frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq U_{(\lceil nt \rceil, n-1)}) - t \right| \leq \frac{1}{n} + \left| \frac{1}{n} \sum_{j=1}^{n-1} \mathbb{1}(U_j \leq U_{(\lceil nt \rceil, n-1)}) - t \right| \leq \frac{2}{n}$$

which shows that t can replace $\frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq U_{(\lceil nt \rceil, n-1)})$ in the convergence (3.6). This establishes (i). Writing

$$|U_{(j,n)} - U_{(j-1,n)}| \leq \left| U_{(j,n)} - \frac{j}{n} \right| + \left| \frac{j}{n} - \frac{j-1}{n} \right| + \left| \frac{j-1}{n} - U_{(j-1,n)} \right|$$

for $n \geq 1$ and $1 \leq j \leq n+1$, we see that (ii) follows immediately from (i).

For $x \in [0, 1]$, we can write

$$\begin{aligned} |\text{Leb}([0, x]) - P_n([0, x])| &= \left| x - \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}(U_j \leq x) \right| \\ &\leq \left| x - \frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq x) \right| + \frac{1}{n} |\mathbb{1}(U_n \leq x) - \mathbb{1}(U_0 \leq x)| \end{aligned}$$

and apply the limit in (3.5) to see that

$$\sup_{x \in [0,1]} |\text{Leb}([0, x]) - P_n([0, x])| \longrightarrow 0$$

as $n \rightarrow \infty$. Using the decomposition

$$\nu((x, y]) = \nu([0, y]) - \nu([0, x])$$

for any probability measure ν on $[0, 1]$ and $x < y$ in $[0, 1]$, we then obtain the convergence

$$\sup_{x < y} |\text{Leb}((x, y]) - P_n((x, y])| \longrightarrow 0$$

as $n \rightarrow \infty$. Recalling that the sequence $(U_j)_{j \geq 1}$ contains distinct values, we have the bound

$$P_n\{x\} = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}(U_j = x) \leq \frac{1}{n}$$

for any $x \in [0, 1]$. In particular, we can add or remove the endpoints of the interval $(x, y]$ above while maintaining the convergence. This establishes the result for $k = 1$. The extension to general k follows from the additivity property of measures. \square

3.4. Useful properties of the relation \prec and the function ϕ . This section is dedicated to analyzing the relation \prec and the function ϕ . The goal is twofold: not only will this allow us to establish that μ_p^{rec} is a permuton a.s., but it will prepare us for the convergence argument. We begin with basic properties of the relation \prec .

LEMMA 3.3. — *Let $x, y, z \in [0, 1]$. Then $i_{x,y}$, $i_{y,z}$, and $i_{x,z}$ can be assigned the labels a , b , and c so that*

$$a = b \leq c.$$

Moreover, if

$$i_{x,y} = i_{y,z} < \infty,$$

then

$$y \prec x, z \quad \text{or} \quad x, z \prec y.$$

Proof. — The intervals $\mathcal{I}_{x,y}$, $\mathcal{I}_{y,z}$, and $\mathcal{I}_{x,z}$ can be assigned the labels A , B , and C so that they satisfy

$$A = B \sqcup C. \quad (3.7)$$

It follows immediately that

$$\begin{aligned} \inf \{j \geq 1 : U_j \in A\} &= \inf \{j \geq 1 : U_j \in B \sqcup C\} \\ &= \min(\inf \{j \geq 1 : U_j \in B\}, \inf \{j \geq 1 : U_j \in C\}), \end{aligned}$$

establishing the first claim. Assume now that $i_{x,y} = i_{y,z} < \infty$, or equivalently,

$$\inf \{j \geq 1 : U_j \in \mathcal{I}_{x,y}\} = \inf \{j \geq 1 : U_j \in \mathcal{I}_{y,z}\} < \infty.$$

Then the intervals $\mathcal{I}_{x,y}$ and $\mathcal{I}_{y,z}$ are not disjoint ($U_{i_{x,y}} \in \mathcal{I}_{x,y} \cap \mathcal{I}_{y,z}$). This implies that y is either the minimum or maximum of $\{x, y, z\}$. In particular, the inequality $(y - z)/(y - x) > 0$ holds. From this, we obtain the equivalence

$$x \prec y \iff (y - x)S_{i_{x,y}} > 0 \iff (y - z)S_{i_{y,z}} > 0 \iff z \prec y.$$

The equivalence

$$y \prec x \iff y \prec z$$

can be obtained similarly. Recalling that $y \neq x$ and $i_{x,y} < \infty$, it follows that $(y - x)S_{i_{x,y}} \neq 0$, so either $x \prec y$ or $y \prec x$. Applying one of the above equivalences concludes the proof. \square

COROLLARY 3.4. — *The relation \prec is a partial order.*

Proof. — Irreflexivity and asymmetry are straightforward to prove, let us consider transitivity. Suppose that $x \prec y$ and $y \prec z$. Since $y \not\prec x$, $z \not\prec y$, and $i_{x,y}, i_{y,z} < \infty$, Lemma 3.3 implies that $i_{x,y} \neq i_{y,z}$. Applying Lemma 3.3 again reveals that $i_{x,z} = \min(i_{x,y}, i_{y,z})$ and $x \prec z$. This shows that \prec is transitive. \square

Now we turn our attention to the function ϕ . For this analysis, we make use of the functions

$$\phi_k(x) = \text{Leb}(\{y \in [0, 1] : y \prec x, i_{x,y} \leq k\}), \quad k \geq 0,$$

which serve as approximations to ϕ . Indeed, the continuity of measure property implies that $\phi_k \nearrow \phi$ pointwise as $k \rightarrow \infty$. In some sense, the function ϕ_k encodes the k^{th} partial order defined in Section 3.1. See Figure 3.2 for an illustration. The basic properties of these functions are summarized in the following result.

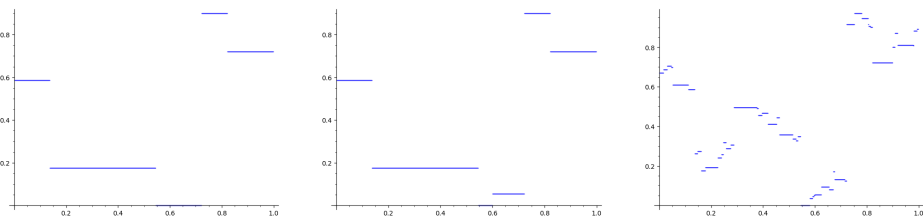


FIGURE 3.2. Graphs of the functions ϕ_k for $k \in \{4, 5, 49\}$, associated with our sampled sequences $(U_j)_{j \leq 5}$ and $(S_j)_{j \leq 6}$, whose first values are given in Eq. (3.1).

PROPOSITION 3.5. — *Let $x, y \in [0, 1]$ and $k \geq 0$. The following statements hold:*

- (i) $i_{x,y} > k \iff \phi_k(x) = \phi_k(y)$,
- (ii) $x \prec y, i_{x,y} \leq k \iff \phi_k(x) < \phi_k(y)$,
- (iii) $\phi_k(x) = \text{Leb}(\{y \in [0, 1] : \phi_k(y) < \phi_k(x)\})$;
- (iv) ϕ_k is constant on each of the intervals

$$[0, U_{(1,k)}), [U_{(1,k)}, U_{(2,k)}), \dots, [U_{(k-1,k)}, U_{(k,k)}), [U_{(k,k)}, 1],$$

and assumes different values on distinct intervals.

Proof. — (i)(\implies) Let $x, y \in [0, 1]$ and $k \geq 0$. Suppose that $i_{x,y} > k$. Let us show that $L_x = L_y$, where

$$L_t = \{s : s \prec t, i_{s,t} \leq k\}, \quad t \in [0, 1].$$

Let $z \in L_x$. This implies $i_{x,z} \leq k$, and in particular $i_{x,z} < i_{x,y}$. From this, Lemma 3.3 gives us that $i_{y,z} = i_{x,z} \leq k$ and $z \prec y$. Therefore, $z \in L_y$, establishing the containment $L_x \subset L_y$. The reverse containment holds by symmetry. Writing $\phi_k(x) = \text{Leb}(L_x) = \text{Leb}(L_y) = \phi_k(y)$ establishes the desired result.

(i) (\impliedby) Suppose now that $i_{x,y} \leq k$. It follows that $x \neq y$ and $S_{i_{x,y}} \neq 0$, so either $x \prec y$ or $y \prec x$. Without loss of generality, we can assume that $x \prec y$. Let us show that $L_x \subset L_y$. Take $z \in L_x$. Since $z \prec x \prec y$, Corollary 3.4 implies that $z \prec y$ and Lemma 3.3 implies that $i_{x,z} \neq i_{x,y}$. Applying Lemma 3.3 again, we find that $i_{y,z} = \min(i_{x,z}, i_{x,y}) \leq k$, and consequently, $z \in L_y$. This establishes the containment $L_x \subset L_y$.

Thus we can write $\phi_k(y) = \phi_k(x) + \text{Leb}(L_y \setminus L_x)$, and it only remains to show that $L_y \setminus L_x$ has positive Lebesgue measure. To this end, we define

$$U^- = \begin{cases} \max\{U_j : -1 \leq j \leq k, U_j \leq x\}, & x \neq 1, \\ \max\{U_j : 0 \leq j \leq k\} & x = 1, \end{cases}$$

and

$$U^+ = \begin{cases} \min\{U_j : -1 \leq j \leq k, x < U_j\}, & x \neq 1, \\ 1 & x = 1, \end{cases}$$

and will show that (U^-, U^+) is a nonempty interval contained in $L_y \setminus L_x$. It should be clear that (U^-, U^+) is nonempty, that $x \in [U^-, U^+)$ (except when $x = 1$) and that

$$(U^-, U^+) \cap \{U_1, \dots, U_k\} = \emptyset.$$

Now take $z \in (U^-, U^+)$. We have $\mathcal{I}_{x,z} \subset (U^-, U^+)$ (except in the case $x = 1$, where U^+ should be included). Recalling that $U_j \in (0, 1)$ for $j \geq 1$, it can be verified that

$$\mathcal{I}_{x,z} \cap \{U_1, \dots, U_k\} \subset (U^-, U^+) \cap \{U_1, \dots, U_k\} = \emptyset.$$

It follows immediately that $i_{x,z} > k$, so $z \notin L_x$. Since $i_{x,y} \leq k$ and $x \prec y$, we have $i_{x,y} < i_{x,z}$ and Lemma 3.3 gives us that $i_{y,z} = i_{x,y} \leq k$ and $z \prec y$, so $z \in L_y$. This establishes the containment $(U^-, U^+) \subset L_y \setminus L_x$, concluding the proof.

(ii) (\implies) This statement was proved in the proof of (i) (\impliedby).

(ii) (\impliedby) Let $x, y \in [0, 1]$ and $k \geq 0$ and suppose that $\phi_k(x) < \phi_k(y)$. Using (i), we have that $i_{x,y} \leq k$. As before, we must have that $x \prec y$ or $y \prec x$. However, the forward implication in (ii) implies that $y \not\prec x$. Therefore, $x \prec y$.

(iii) This claim follows from the definition of ϕ_k and (ii)

(iv) This claim follows immediately from (i). \square

The above result leads to the following estimate for ϕ , which plays a crucial role in the convergence argument.

COROLLARY 3.6. — *Let $k \geq 1$ and ν be a probability measure on $[0, 1]$. The following inequality holds:*

$$\sup_{x \in [0,1]} |\nu\{y : y \prec x\} - \nu\{y : \phi_k(y) < \phi_k(x)\}| \leq \Delta_k + \sup_{J \in \mathcal{A}_1} |\text{Leb}(J) - \nu(J)|. \quad (\text{i})$$

Consequently, the following convergence holds on \mathcal{E} :

$$\sup_{x \in [0,1]} |\phi(x) - P_n\{y : y \prec x\}| \longrightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{ii})$$

Proof. — Let $k \geq 1$ and ν be a probability measure on $[0, 1]$. Using Proposition 3.5, we have that

$$\begin{aligned} |\nu\{y : y \prec x\} - \nu\{y : \phi_k(y) < \phi_k(x)\}| &= \nu\{y : y \prec x, i_{x,y} > k\} \\ &\leq \nu\{y : i_{x,y} > k\} = \nu\{y : \phi_k(y) = \phi_k(x)\} = \nu(\phi_k^{-1}(\phi_k(x))). \end{aligned}$$

From Proposition 3.5 (iv), $\phi_k^{-1}(\phi_k(x))$, is an interval between two consecutive values in the set $\{U_{-1}, U_0, \dots, U_k\}$. Therefore

$$\begin{aligned} \text{Leb}(\phi_k^{-1}(\phi_k(x))) &\leq \Delta_k \text{ and } |\nu(\phi_k^{-1}(\phi_k(x))) - \text{Leb}(\phi_k^{-1}(\phi_k(x)))| \\ &\leq \sup_{J \in \mathcal{A}_1} |\text{Leb}(J) - \nu(J)|, \end{aligned}$$

establishing (i). Taking $\nu = \text{Leb}$ and $\nu = P_n$ in (i) gives us that, for $k \geq 1$,

$$\sup_{x \in [0,1]} |\text{Leb}\{y : y \prec x\} - \text{Leb}\{y : \phi_k(y) < \phi_k(x)\}| \leq \Delta_k \quad (3.8)$$

and, for $n, k \geq 1$,

$$\sup_{x \in [0,1]} |P_n\{y : y \prec x\} - P_n\{y : \phi_k(y) < \phi_k(x)\}| \leq \Delta_k + \sup_{J \in \mathcal{A}_1} |\text{Leb}(J) - P_n(J)|.$$

Note that \mathcal{A}_1 can be replaced by \mathcal{A}_k above since $\mathcal{A}_1 \subset \mathcal{A}_k$ for all $k \geq 1$. Observe also that the set $\{y : \phi_k(y) < \phi_k(x)\}$ lies in \mathcal{A}_k for every $x \in [0, 1]$ (see Proposition 3.5 (iv)), from which we obtain the following inequality (for $n, k \geq 1$):

$$\sup_{x \in [0,1]} |\text{Leb}\{y : \phi_k(y) < \phi_k(x)\} - P_n\{y : \phi_k(y) < \phi_k(x)\}| \leq \sup_{J \in \mathcal{A}_k} |\text{Leb}(J) - P_n(J)|.$$

Collecting these bounds, we obtain the estimates (again for $n, k \geq 1$)

$$\sup_{x \in [0,1]} |\text{Leb}\{y : y \prec x\} - P_n\{y : y \prec x\}| \leq 2\Delta_k + 2 \sup_{J \in \mathcal{A}_k} |\text{Leb}(J) - P_n(J)|.$$

Since $\text{Leb}\{y : y \prec x\} = \phi(x)$, applying Proposition 3.2 (items (ii) and (iii)) concludes the proof. \square

Finally, we establish the main properties of the objects \prec , ϕ , and μ_p^{rec} .

PROPOSITION 3.7. — *On \mathcal{E} , the following statements hold:*

- (i) *the relation \prec is a total order,*
- (ii) *$\phi_k \rightarrow \phi$ uniformly on $[0, 1]$ as $k \rightarrow \infty$,*
- (iii) *ϕ is continuous on $[0, 1] \setminus \{U_j\}_{j \geq 1}$,*
- (iv) *ϕ preserves the measure Leb , and thus μ_ϕ is a permuton.*

Proof. — Take $x \neq y$ in $[0, 1]$ and suppose that \mathcal{E} occurs. From Proposition 3.2, we know that the maximal gap Δ_k of the sequence $(U_j)_{j \leq k}$ tends to 0, so the infinite sequence $(U_j)_{j \geq 1}$ must intersect the nonempty interval $\mathcal{I}_{x,y}$. It follows that x and y are comparable in the order \prec . This establishes (i).

The uniform convergence in (ii) follows immediately from Proposition 3.5(iii), Eq. (3.8) and Proposition 3.2(ii). The continuity in (iii) then follows from the uniform convergence in (ii) and from Proposition 3.5(iv), which describes the continuity of each ϕ_k .

It remains to prove (iv), i.e. that ϕ preserves the Lebesgue measure. Let $b \in (0, 1)$. From Proposition 3.5(iii) and (iv), we see that each ϕ_k attains only finitely many values and that the gap between consecutive values is bounded by Δ_k . In particular, defining y_k such that $\phi_k(y_k)$ is the smallest element in the range of ϕ_k above b (which exists for large k), we have $\lim_{k \rightarrow +\infty} \phi_k(y_k) = b$. Also, using Proposition 3.5(iii), we have

$$\phi_k(y_k) = \text{Leb}(\{z : \phi_k(z) < \phi_k(y_k)\}) = \text{Leb}(\{z : \phi_k(z) \leq b\}).$$

Recalling that $\phi_k \nearrow \phi$, we have that

$$\begin{aligned} (\phi_{\#} \text{Leb})([0, b]) &= \text{Leb}(\{z : \phi(z) \leq b\}) = \lim_{k \rightarrow \infty} \text{Leb}(\{z : \phi_k(z) \leq b\}) \\ &= \lim_{k \rightarrow \infty} \phi_k(y_k) = b. \end{aligned}$$

Since this holds for every $b \in (0, 1)$, the measures $\phi_{\#} \text{Leb}$ and Leb are the same, and (iv) holds. \square

3.5. The permutations. We continue our construction by realizing the recursive separable permutations on our probability space. As mentioned earlier, the strategy will be to identify the partial orders from Section 3.1 as permutations. Informally, we do this by viewing the intervals in a partial order as forming a “shape” that defines a permutation diagram. For example, the 2nd partial order illustrated in Figure 3.1 has the shape of the permutation 132 since its leftmost interval $[0, U_1]$ is placed at the lowest level, its middle interval $[U_1, U_2]$ is placed at the highest level, and its rightmost interval $[U_2, 1]$ is placed at an intermediate level. Note that this shape can also be deduced from simply considering the left endpoints $0, U_1$, and U_2 , which satisfy $0 < U_1 < U_2$ and $0 \prec U_2 \prec U_1$.

Throughout this section, we will assume that the $(U_j)_{j \geq 1}$ are distinct and different from 0 and 1. This ensures that the random set $E_n = \{U_j, 0 \leq j \leq n-1\}$ has size n . Equipping each E_n with the natural order $<$ and the random order \prec , the above considerations prompt us to define

$$\lambda_n = \text{Perm}(E_n, <, \prec), \quad n \geq 1. \quad (3.9)$$

An illustration is provided in Figure 3.3, which should be compared with Figures 3.1 and 3.2.

In Proposition 3.9, we confirm that this permutation sequence has the same distribution as $(\sigma^{(n),p})_{n \geq 1}$. The proof uses some standard exchangeability properties of i.i.d. random variables, implying for example that the relative order of the $(U_i)_{i \leq n}$ is independent from the set of their values $\{U_i, i \leq n\}$. As a first step, we establish a recursion for λ_n , which will require some notation.

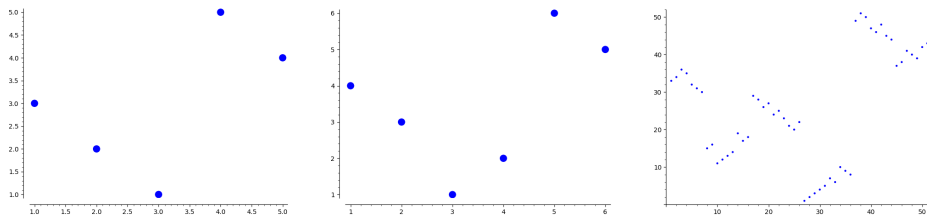


FIGURE 3.3. The permutations λ_n for $n \in \{5, 6, 50\}$, associated with our sampled sequences $(U_j)_{j \leq 5}$ and $(S_j)_{j \leq 6}$, whose first values are given in Eq. (3.1).

Given a permutation τ of n , a sign $s \in \{\oplus, \ominus\}$, and an integer $k \in \{1, \dots, n\}$, let $\tau_{s,k}$ be the permutation obtained by writing τ in one-line notation, increasing all values bigger than $j = \tau(k)$, and replacing j by $j + 1$ if $s = \oplus$ or by $j - 1$ if $s = \ominus$. For a permutation σ of n and an integer k , we denote by $\sigma^{\downarrow k}$ the pattern induced by σ on the set of positions $\{1, \dots, n\} \setminus \{k\}$ (in other words, we erase σ_k in the one-line notation and decrease by 1 all values bigger than σ_k to get a permutation).

With this notation in hand, it can be verified that $\tau_{s,k}$ is the unique permutation σ of $n + 1$ satisfying the following properties:

- (i) $\sigma^{\downarrow k}$ and $\sigma^{\downarrow(k+1)}$ are both equal to τ ;
- (ii) $\sigma(k + 1) > \sigma(k)$ if $s = \oplus$ (resp. $\sigma(k + 1) < \sigma(k)$ if $s = \ominus$).

PROPOSITION 3.8. — Suppose that the elements $(U_j)_{j=1}^n$ are distinct. Let R_n denote the rank of U_n in $\{U_j\}_{j=1}^n$ (i.e. the unique integer satisfying $U_{(R_n, n)} = U_n$). The following recursion holds:

$$\lambda_{n+1} = (\lambda_n)_{S_n, R_n}, \quad n \geq 1.$$

Before proving the result, let us consider an example using the sampled values $(U_j)_{j \leq 5}$ and $(S_j)_{j \leq 5}$ in Eq. (3.1). Taking $n = 5$, the relevant quantities are $S_5 = \oplus$, $R_5 = 3$, and the permutations λ_5 and λ_6 depicted in Figure 3.3. It can be seen from the diagrams that the identity $\lambda_6 = (\lambda_5)_{\oplus, 3}$ does hold: λ_6 is obtained from λ_5 by an increasing inflation of its third point from the left.

Proof. — We will verify that λ_{n+1} satisfies the three conditions that characterize $(\lambda_n)_{S_n, R_n}$. Writing $k = R_n$, we note that U_n corresponds to the $(k + 1)^{\text{st}}$ point from the left in λ_{n+1} (note that the index j starts from 0 in eq. (3.9)). Therefore we have

$$(\lambda_{n+1})^{\downarrow k+1} = \text{Perm}(\{U_j, 0 \leq j \leq n - 1\}, <, \prec) = \lambda_n.$$

Let us consider the pattern $(\lambda_{n+1})^{\downarrow k}$. The k^{th} point from the left in λ_{n+1} corresponds to $U_{(k-1, n)}$, so that

$$(\lambda_{n+1})^{\downarrow k} = \text{Perm}(\{U_j, 0 \leq j \leq n\} \setminus \{U_{(k-1, n)}\}, <, \prec).$$

To show that this is also λ_n , we need to show that U_n and $U_{(k-1, n)}$ compare in the same way with other U_j 's ($j \leq n$), both for the natural order $<$ and for the random order \prec .

The case of the natural order $<$ is trivial by definition of the ordered statistics, since $U_n = U_{(k, n)}$. Consider the random order \prec . Since $U_n = U_{(k, n)}$, the element

U_n is the U with smallest index in the interval $(U_{(k-1,n)}, U_n]$, i.e. $i_{U_{(k-1,n)}, U_n} = n$. On the other hand, for $j < n$ and $U_j \neq U_{(k-1,n)}$, the interval $\mathcal{I}_{U_{(k-1,n)}, U_j}$ contains either U_j or $U_{(k-1,n)}$, so that $i_{U_{(k-1,n)}, U_j} < n$. Applying Lemma 3.3, we then find that $i_{U_{(k-1,n)}, U_j} = i_{U_j, U_n} < \infty$, and $U_{(k-1,n)}$ and U_n compare in the same way with U_j in the order \prec , which is what we needed to show. Thus λ_{n+1} verifies the first condition in the characterization of $(\lambda_n)_{S_n, R_n}$.

It remains to show that $\lambda_{n+1}(k) < \lambda_{n+1}(k+1)$ if $S_n = \oplus$, (resp. $\lambda_{n+1}(k) > \lambda_{n+1}(k+1)$ if $S_n = \ominus$). We consider the case $S_n = \oplus$. Since the k^{th} and $(k+1)^{\text{st}}$ points from the left in λ_{n+1} correspond to $U_{(k-1,n)}$ and U_n respectively, it suffices to show that $U_{(k-1,n)} \prec U_n$. This follows from the fact that $i_{U_{(k-1,n)}, U_n} = n$ (see above) and the assumption $S_n = \oplus$ (recall that $U_{(k-1,n)} < U_n$ for the natural order).

We have proved that λ_{n+1} verifies the second condition in the characterization of $(\lambda_n)_{S_n, R_n}$, implying $\lambda_{n+1} = (\lambda_n)_{S_n, R_n}$, as desired. \square

PROPOSITION 3.9. — *The permutations $(\lambda_n)_{n \geq 1}$ are recursive separable permutations. In other words,*

$$(\lambda_n)_{n \geq 1} \stackrel{d}{=} (\sigma^{(n), p})_{n \geq 1}.$$

Proof. — We start with the following observation.

- To compare U_j and U_k in the order \prec , we need to look at the sign $S_{i_{U_j, U_k}}$. If $j, k < n$ and $U_j \neq U_k$, then the interval \mathcal{I}_{U_j, U_k} contains either U_j or U_k , and thus $i_{U_j, U_k} < n$. Therefore, the restriction of \prec to $\{U_0, \dots, U_{n-1}\}$ only depends on the restriction of the natural order $<$ to $\{U_0, \dots, U_{n-1}\}$ and on the signs S_1, \dots, S_{n-1} . Consequently, the tuple of permutations $(\lambda_1, \dots, \lambda_n)$ only depends on this data as well.
- On the other hand, R_n describes how U_n fits into the gaps of the set $\{U_0, \dots, U_{n-1}\}$, and is independent of its order structure.

Therefore the random variables $(\lambda_1, \dots, \lambda_n)$, S_n , and R_n are independent. Combining this with Proposition 3.8, this implies that $(\lambda_n)_{n \geq 1}$ is a Markov process and we can compute its transition kernel. If π and τ are permutations of n and $n+1$ respectively, one has

$$\begin{aligned} \mathbb{P}(\lambda_{n+1} = \tau \mid \lambda_n = \pi) &= \sum_{j=1}^n \sum_{s \in \{\oplus, \ominus\}} \mathbb{1}(\pi_{s,j} = \tau) \mathbb{P}(R_n = j, S_n = s) \\ &= \sum_{j=1}^n \left(\frac{p}{n} \mathbb{1}(\pi_{\oplus, j} = \tau) + \frac{1-p}{n} \mathbb{1}(\pi_{\ominus, j} = \tau) \right) \\ &= \mathbb{P}(\sigma^{(n+1), p} = \tau \mid \sigma^{(n), p} = \pi) \end{aligned}$$

Since $(\lambda_n)_{n \geq 1}$ and $(\sigma^{(n), p})_{n \geq 1}$ are both Markov chains with the same transition kernel and the same initial distribution (λ_1 and $\sigma^{(1), p}$ are both equal to the unique permutation of 1 a.s.), they have the same distribution. \square

The following result is the last one in this section. Here, we show that our permutation model satisfies another type of recursion, referred to as *consistency*. It is not needed for the convergence argument, but it is useful in the next section for studying the expected pattern densities of μ_p^{rec} .

PROPOSITION 3.10. — *The sequence $(\lambda_n)_{n \geq 1}$ is a consistent family of random permutations, in the sense of [6, Definition 2.8]: namely, for $n \geq 1$, the permutation obtained by removing a uniformly random point from λ_{n+1} is distributed as λ_n .*

Proof. — To avoid repetition, let us take the following convention throughout the proof: each new random variable introduced in this proof should be assumed to be independent of all previously defined random variables.

The $n = 1$ case of the proposition is trivial. Proceeding by induction, we fix $n > 1$ and assume that the result holds for $n - 1$. Let j be a uniformly random integer in $\{1, \dots, n + 1\}$ and A be the event $\{R_n \leq j \leq R_n + 1\}$. As we will show later, the following statements hold:

- (i) λ_n is independent of A ,
- (ii) $\lambda_{n+1}^{\downarrow j} = \lambda_n$ on A , and
- (iii) conditionally given A^c , $\lambda_{n+1}^{\downarrow j}$ is distributed as λ_n .

The result follows from these statements. Indeed, if τ is a permutation of n , then

$$\mathbb{P}\left(\lambda_{n+1}^{\downarrow j} = \tau\right) = \mathbb{P}\left(\lambda_{n+1}^{\downarrow j} = \tau | A^c\right) \mathbb{P}(A^c) + \mathbb{P}\left(\lambda_{n+1}^{\downarrow j} = \tau | A\right) \mathbb{P}(A).$$

Provided that (iii) holds, we have $\mathbb{P}(\lambda_{n+1}^{\downarrow j} = \tau | A^c) = \mathbb{P}(\lambda_n = \tau)$, while (i), (ii) would imply $\mathbb{P}(\lambda_{n+1}^{\downarrow j} = \tau | A) = \mathbb{P}(\lambda_n = \tau | A) = \mathbb{P}(\lambda_n = \tau)$. Therefore, given (i), (ii), (iii), one has

$$\mathbb{P}\left(\lambda_{n+1}^{\downarrow j} = \tau\right) = \mathbb{P}(\lambda_n = \tau) \mathbb{P}(A^c) + \mathbb{P}(\lambda_n = \tau) \mathbb{P}(A) = \mathbb{P}(\lambda_n = \tau).$$

It remains, then, to verify the above statements. As explained in the proof of Proposition 3.9, λ_n is independent from R_n . Moreover, by construction, j is independent from (λ_n, R_n) . Thus λ_n is independent from (j, R_n) and hence from the event A , proving (i).

Moreover it follows from Proposition 3.8 that, a.s., $\lambda_{n+1}^{\downarrow R_n} = \lambda_{n+1}^{\downarrow R_n+1} = \lambda_n$, implying (ii). For the third statement, we introduce the random variables

$$\begin{aligned} X &= j - \mathbb{1}(j > R_n), \\ Y &= R_n - \mathbb{1}(R_n > j). \end{aligned}$$

Using Proposition 3.8 and an obvious commutation relation between the operator $\downarrow j$ and $(\cdot)_{s,k}$, we have

$$\lambda_{n+1}^{\downarrow j} = ((\lambda_n)_{S_n, R_n})^{\downarrow j} = (\lambda_n^{\downarrow X})_{S_n, Y}, \quad \text{on } A^c.$$

Notice also that S_n , λ_n , X , and Y are mutually independent given A^c , S_n and λ_n are independent of A^c , X is uniformly distributed in $\{1, \dots, n\}$ conditionally given A^c , and Y is uniformly distributed in $\{1, \dots, n - 1\}$ conditionally given A^c . Therefore, the conditional distribution of (S_n, λ_n, X, Y) given A^c is exactly the distribution of $(S_n, \lambda_n, k, \ell)$, where k is a uniformly random integer in $\{1, \dots, n\}$ and ℓ is a uniformly random integer in $\{1, \dots, n - 1\}$. Recalling from Proposition 3.9 that λ_{n-1} , S_{n-1} , and R_{n-1} are mutually independent and making use of the induction hypothesis, we find that conditionally given A^c ,

$$\lambda_{n+1}^{\downarrow j} = (\lambda_n^{\downarrow X})_{S_n, Y} \stackrel{d}{=} (\lambda_n^{\downarrow k})_{S_n, \ell} \stackrel{d}{=} (\lambda_{n-1})_{S_{n-1}, \ell} \stackrel{d}{=} \lambda_n.$$

This establishes the third fact and concludes the proof. \square

3.6. The convergence argument.

Proof of Theorem 1.2. — We recall that the event

$$\mathcal{E} = \left\{ (U_j)_{j \geq 1} \text{ are distinct, } \sup_{x \in [0,1]} \left| x - \frac{1}{n} \sum_{j=1}^n \mathbb{1}(U_j \leq x) \right| \xrightarrow[n \rightarrow \infty]{} 0 \right\}$$

holds almost surely. Also, since the permutations $(\lambda_n)_{n \geq 1}$ are recursive separable permutations (see Proposition 3.9),

it suffices to show the permuton convergence

$$\mu_{\lambda_n} \xrightarrow[n \rightarrow \infty]{} \mu_p^{\text{rec}} \quad \text{on } \mathcal{E}.$$

Making use of (2.1), the identity $\mu_p^{\text{rec}} = (\cdot, \phi(\cdot))_{\#} \text{Leb}$, and Proposition 2.1, we can reformulate this convergence as the convergence of functions

$$f_{\lambda_n} \longrightarrow \phi \quad \text{in } L^1[0,1] \quad \text{on } \mathcal{E}.$$

We will establish this convergence by showing that $f_{\lambda_n} \rightarrow \phi$ pointwise almost everywhere in $[0,1]$ whenever \mathcal{E} occurs.

To this end, fix $x \in (0,1] \setminus \{U_j\}_{j \geq 1}$ and suppose that \mathcal{E} occurs. Since $x > 0$, the quantity $\lceil nx \rceil$ lies in $\{1, \dots, n\}$ and we can write

$$\begin{aligned} |f_{\lambda_n}(x) - \phi(x)| &\leq |f_{\lambda_n}(x) - P_n(\{y : y \prec U_{(\lceil nx \rceil - 1, n-1)}\})| \\ &\quad + |P_n(\{y : y \prec U_{(\lceil nx \rceil - 1, n-1)}\}) - \phi(U_{(\lceil nx \rceil - 1, n-1)})| \\ &\quad + |\phi(U_{(\lceil nx \rceil - 1, n-1)}) - \phi(x)|. \end{aligned} \quad (3.10)$$

Corollary 3.6(ii) tells us that the second term above will converge to zero as $n \rightarrow \infty$. The third term also converges to zero: indeed, Proposition 3.2 implies that

$$\begin{aligned} |x - U_{(\lceil nx \rceil - 1, n-1)}| &\leq |x - U_{(\lceil nx \rceil, n-1)}| + |U_{(\lceil nx \rceil, n-1)} - U_{(\lceil nx \rceil - 1, n-1)}| \\ &\leq |x - U_{(\lceil nx \rceil, n-1)}| + \Delta_{n-1} \\ &\xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

and ϕ is continuous at x (see Proposition 3.7(iii)). It only remains, then, to show that the first term in the upper bound (3.10) also converges to zero. This follows from the definition of λ_n and Eq. (2.3), which allows us to write

$$\begin{aligned} f_{\lambda_n}(x) &= \frac{\lambda_n(\lceil nx \rceil)}{n} = \frac{1}{n} + \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}(U_j \prec U_{(\lceil nx \rceil - 1, n-1)}) \\ &= \frac{1}{n} + P_n(\{y : y \prec U_{(\lceil nx \rceil - 1, n-1)}\}). \end{aligned}$$

Hence, whenever \mathcal{E} occurs, for any $x \in (0,1] \setminus \{U_j\}_{j \geq 1}$, the quantity $f_{\lambda_n}(x)$ tends to $\phi(x)$. This concludes the proof. \square

4. PROPERTIES OF THE RECURSIVE SEPARABLE PERMUTON

4.1. Self-similarity. Fix $p \in (0,1)$. We recall informally the definition of Φ_p from the introduction. If μ is a random permuton, then $\Phi_p(\mu)$ is the random permuton obtained as follows.

- We let U and S be independent random variables, U being uniform in $[0,1]$ and S being \oplus with probability p and \ominus with probability $1-p$.

- We construct $\Phi_p(\boldsymbol{\mu})$ by juxtaposing two independent copies $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ of $\boldsymbol{\mu}$, where $\boldsymbol{\mu}_0$ is scaled by a factor U , while $\boldsymbol{\mu}_1$ is scaled by a factor $1 - U$. If $S = \oplus$, we put $\boldsymbol{\mu}_0$ below and on the left of $\boldsymbol{\mu}_1$, while, if $S = \ominus$, we put $\boldsymbol{\mu}_0$ above and on the left of $\boldsymbol{\mu}_1$.

The goal of this section is to prove Proposition 1.6, which states that the law of $\boldsymbol{\mu}_p^{\text{rec}}$ is the unique probability distribution invariant by Φ_p .

We first prove the uniqueness of such an invariant probability distribution. For this, we see, with a small abuse of notation, Φ_p as a map from the set $\mathcal{M}_1(\mathcal{P})$ of probability measures on the set \mathcal{P} of permutons to itself. Both \mathcal{P} and $\mathcal{M}_1(\mathcal{P})$ are endowed with the first Wasserstein distance; to avoid confusion, we will use a boldface notation \mathbf{d}_W for the distance on $\mathcal{M}_1(\mathcal{P})$ and a standard d_W for that on \mathcal{P} .

LEMMA 4.1. — *The map $\Phi_p : \mathcal{M}_1(\mathcal{P}) \rightarrow \mathcal{M}_1(\mathcal{P})$ is a contraction with Lipschitz constant at most $2/3$.*

Proof. — Let \mathbb{P} and \mathbb{Q} be two probability measures on the set \mathcal{P} of permutons and call $d := \mathbf{d}_W(\mathbb{P}, \mathbb{Q})$ their Wasserstein distance. Fix $\varepsilon > 0$. By definition of Wasserstein distance, one can find random permutons $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ on the same probability space such that $\boldsymbol{\mu}$ has distribution \mathbb{P} , $\boldsymbol{\nu}$ has distribution \mathbb{Q} and

$$\mathbb{E}(d_W(\boldsymbol{\mu}, \boldsymbol{\nu})) < d + \varepsilon.$$

We consider two independent copies $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0)$ and $(\boldsymbol{\mu}_1, \boldsymbol{\nu}_1)$ of the pair $(\boldsymbol{\mu}, \boldsymbol{\nu})$. We also consider a single pair (U, S) , where U is a uniform random variable in $[0, 1]$ and S a random sign with $\mathbb{P}(S = \oplus) = p$, such that U and S are independent from each other and from $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0)$ and $(\boldsymbol{\mu}_1, \boldsymbol{\nu}_1)$. By definition of Φ_p , the measure $\Phi_p(\mathbb{P})$ (resp. $\Phi_p(\mathbb{Q})$) is the distribution of the random permuton $\boldsymbol{\mu}_0 \otimes_{(U,S)} \boldsymbol{\mu}_1$ (resp. $\boldsymbol{\nu}_0 \otimes_{(U,S)} \boldsymbol{\nu}_1$). Therefore

$$\mathbf{d}_W(\Phi_p(\mathbb{P}), \Phi_p(\mathbb{Q})) \leq \mathbb{E}(d_W(\boldsymbol{\mu}_0 \otimes_{(U,S)} \boldsymbol{\mu}_1, \boldsymbol{\nu}_0 \otimes_{(U,S)} \boldsymbol{\nu}_1)). \quad (4.1)$$

It is straightforward to see that, a.s.,

$$d_W(\boldsymbol{\mu}_0 \otimes_{(U,S)} \boldsymbol{\mu}_1, \boldsymbol{\nu}_0 \otimes_{(U,S)} \boldsymbol{\nu}_1) \leq U^2 d_W(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0) + (1 - U)^2 d_W(\boldsymbol{\mu}_1, \boldsymbol{\nu}_1), \quad (4.2)$$

where the factor U^2 is explained by the rescaling of distances and of weights of $\boldsymbol{\mu}_0$, both by a factor U in the construction of $\boldsymbol{\mu}_0 \otimes_{(U,S)} \boldsymbol{\mu}_1$ (and similarly the factor $(1 - U)^2$ comes from the rescaling of distances and weights in $\boldsymbol{\mu}_1$ by $1 - U$). From Eqs. (4.1) and (4.2), using the independence of U from $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0)$ and $(\boldsymbol{\mu}_1, \boldsymbol{\nu}_1)$ and the equality $(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0) \stackrel{d}{=} (\boldsymbol{\mu}_1, \boldsymbol{\nu}_1)$, we get

$$\mathbf{d}_W(\Phi_p(\mathbb{P}), \Phi_p(\mathbb{Q})) \leq \mathbb{E}(U^2 + (1 - U)^2) \mathbb{E}(d_W(\boldsymbol{\mu}_0, \boldsymbol{\nu}_0)) < \frac{2}{3}(d + \varepsilon).$$

Since this holds for any $\varepsilon > 0$, we have

$$\mathbf{d}_W(\Phi_p(\mathbb{P}), \Phi_p(\mathbb{Q})) \leq \frac{2}{3} d = \frac{2}{3} \mathbf{d}_W(\mathbb{P}, \mathbb{Q}).$$

This proves the lemma. \square

Lemma 4.1 implies in particular the existence and uniqueness of a probability measure \mathbb{P}_p on the set of permutons such that $\mathbb{P}_p = \Phi_p(\mathbb{P}_p)$. We still need to identify \mathbb{P}_p with the distribution of the recursive separable permuton $\boldsymbol{\mu}_p^{\text{rec}}$. To do this, we first show that the recursive separable *permutations* exhibit self-similarity and then use Theorem 1.2 to carry this property over to the limit. It might be

possible to establish the self-similarity of the permuton directly, but we think that the self-similarity of the permutations is of its own interest.

Below, we denote the distribution of a random variable X by $\text{Law}(X)$. In addition, we use the notion of skew sums and direct sums of permutations introduced in Section 1.3.2.

PROPOSITION 4.2. — *Let $(\sigma^{(n),p})_{n \geq 1}$, $(\tau^{(n),p})_{n \geq 1}$ and $(\rho^{(n),p})_{n \geq 1}$ be independent copies of the permutation process defined in Section 1.1, and let I be a uniform integer in $\{1, \dots, n-1\}$. Then, for any fixed $n \geq 1$, we have*

$$\text{Law}(\sigma^{(n),p}) = p \text{Law}\left(\tau^{(I),p} \oplus \rho^{(n-I),p}\right) + (1-p) \text{Law}\left(\tau^{(I),p} \ominus \rho^{(n-I),p}\right).$$

We note that this is an equality for fixed $n \geq 1$, and it does not extend to a recursive description of the law of the process $(\sigma^{(n),p})_{n \geq 1}$.

Proof. — For $n = 2$, we have $\sigma^{(2),p} = 12$ with probability p and $\sigma^{(2),p} = 21$ with probability $1-p$. We will show that conditionally on $\sigma^{(n),p} = 12$, we have

$$\text{Law}\left(\sigma^{(n),p} \mid \sigma^{(2),p} = 12\right) = \text{Law}\left(\tau^{(I),p} \oplus \rho^{(n-I),p}\right). \quad (4.3)$$

We recall that $(\sigma^{(n),p})_{n \geq 1}$ is defined recursively via inflation operations, where one point is replaced by two adjacent points (either in ascending or descending positions). In such operations, we will think at the two new points as the “children” of the point they replace. This defines an ascendant/descendant relation on the set of points of all $\sigma^{(n),p}$ for $n \geq 1$.

In particular, points in $\sigma^{(n),p}$ can be split in two parts, defined as the descendants of the points 1 and 2 in $\sigma^{(2),p}$. Let us call τ , resp. ρ , the pattern formed the descendants of 1, resp. 2. Since points are always replaced by pairs of adjacent points, when $\sigma^{(2),p} = 12$, we have $\sigma^{(n),p} = \tau \oplus \rho$. Conditionally of its size, which we call k , the permutation τ has the same distribution as $\tau^{(k),p}$ since it is obtained from 1 by successive random inflations. Similarly, ρ has the same distribution as $\rho^{(n-k),p}$. Moreover, both are independent conditionally on k .

Therefore, the only remaining thing to be proven in order to establish (4.3) is that k is uniformly distributed in $\{1, \dots, n-1\}$. Letting I_j being the number of descendants of 1 in $\sigma^{(j),p}$, we see that $(I_j)_{j \geq 1}$ has the Markov property and

$$\begin{cases} \mathbb{P}(I_j = k \mid I_{j-1} = k-1) = \frac{k-1}{j-1} \\ \mathbb{P}(I_j = k \mid I_{j-1} = k) = \frac{j-1-k}{j-1} \end{cases}$$

Using this and the base case $I_2 = 1$ a.s., an immediate induction shows that, for any $j \geq 2$, the random variable I_j is uniformly distributed in $\{1, \dots, j-1\}$ (we remark that this is a basic model of Pólya urn.) This concludes the proof of Eq. (4.3).

With similar arguments, one can prove

$$\text{Law}\left(\sigma^{(n),p} \mid \sigma^{(2),p} = 21\right) \stackrel{d}{=} \text{Law}\left(\tau^{(I),p} \ominus \rho^{(n-I),p}\right). \quad (4.4)$$

Since $\mathbb{P}(\sigma^{(n),p} = 12) = p = 1 - \mathbb{P}(\sigma^{(n),p} = 21)$, the proposition follows from Eqs. (4.3) and (4.4). \square

Proof of Proposition 1.6. — Taking the limit $n \rightarrow +\infty$ in Proposition 4.2 gives $\mu_p^{\text{rec}} \stackrel{d}{=} \Phi_p(\mu_p^{\text{rec}})$ (recall from Theorem 1.2 that $\sigma^{(n),p}$ converge a.s., and hence in

distribution to μ_p^{rec}). The uniqueness statement in Proposition 1.6 follows from Lemma 4.1. \square

4.2. Expected pattern densities.

Proof of Proposition 1.7. — From Proposition 3.10, we know that $(\sigma^{(n),p})_{n \geq 1}$ is a consistent family of random permutations. By [6, Proposition 2.9], there exists a random permuton μ such that $\sigma^{(n),p}$ converge to μ in distribution and, for each $n \geq 1$, the random permutation $\text{Sample}(\mu, n)$ has the same distribution as $\sigma^{(n),p}$. On the other hand, $\sigma^{(n),p}$ converges almost surely to μ_p^{rec} (Theorem 1.2), so μ and μ_p^{rec} must be equal in distribution. Therefore,

$$\sigma^{(n),p} \stackrel{d}{=} \text{Sample}(\mu, n) \stackrel{d}{=} \text{Sample}(\mu_p^{\text{rec}}, n), \quad n \geq 1.$$

Together with [6, Theorem 2.5], this implies that

$$\mathbb{E} [\text{dens}(\pi, \mu_p^{\text{rec}})] = \mathbb{P} [\text{Sample}(\mu_p^{\text{rec}}, n) = \pi] = \mathbb{P} [\sigma^{(n),p} = \pi], \quad \pi \in S_n.$$

This proves the first part of Proposition 1.7.

For the second part, we associate with a realization of the random sequence $(\sigma^{(k),p})_{k \leq n}$ a (rooted binary increasing decorated) tree T_n such that $\text{Perm}(T_n) = \sigma^{(n),p}$ (where we use the map Perm from trees to permutations introduced in Section 1.3.2). We proceed by induction on n . For $n = 1$, the tree T_1 is reduced to a single leaf. For $n \geq 1$, assume that T_n is constructed and that in the sampling process for $\sigma^{(n+1),p}$ (Section 1.1), we have chosen some integer j and some sign s (where $s = \oplus$ means that j is replaced by $j, j+1$ and $s = \ominus$ means that j is replaced by $j+1, j$). Then T_{n+1} is obtained by replacing the j^{th} leaf of T_n by an internal node with label $n+1$ and decoration s , with two children which are leaves. It can be verified, using the induction hypothesis $\text{Perm}(T_n) = \sigma^{(n),p}$, that $\text{Perm}(T_{n+1}) = \sigma^{(n+1),p}$. Informally, T_n encodes the history of the construction of $\sigma^{(n),p}$. An example is given on Figure 4.1.

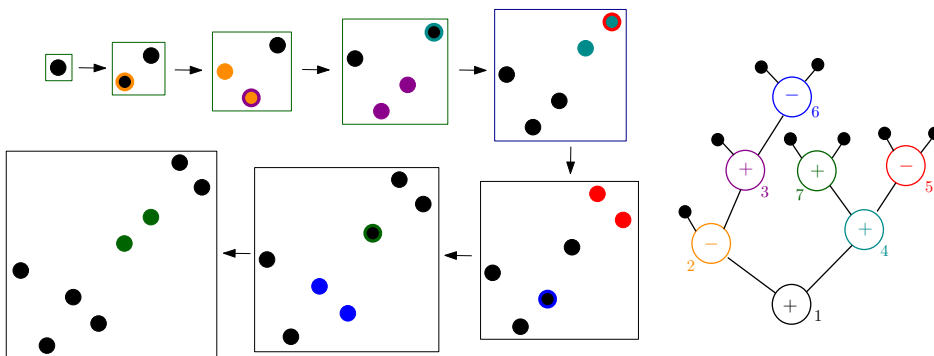


FIGURE 4.1. A realization of the process $(\sigma^{(k),p})_{k \leq 8}$ and the associated tree T_8 . Colors indicate points which have been selected for inflation at each step, and the internal node created in the corresponding step in the construction of T_n .

The above construction yields a random tree T_n living in the same probability space as $(\sigma^{(k),p})_{k \leq n}$. For a fixed (rooted binary increasing decorated) tree T , the event $T_n = T$ amounts to making specific choices of integers j and signs s at each step of the construction of $(\sigma^{(k),p})_{k \leq n}$. Thus we have

$$\mathbb{P}[T_n = T] = \frac{1}{(n-1)!} p^{\oplus(T)} (1-p)^{\ominus(T)},$$

where $\oplus(T)$ and $\ominus(T)$ denote the number of \oplus decorations and \ominus decorations in T respectively. Since $\text{Perm}(T_n) = \sigma^{(n),p}$, this implies that for a given permutation π of n ,

$$\mathbb{P}[\sigma^{(n),p} = \pi] = \frac{1}{(n-1)!} \sum_{T: \text{Perm}(T)=\pi} p^{\oplus(T)} (1-p)^{\ominus(T)}.$$

But an immediate inductive argument shows that whenever $\text{Perm}(T) = \sigma$, we have $\ominus(T) = \text{des}(\sigma)$, and consequently $\oplus(T) = n-1 - \text{des}(\sigma)$. Therefore, all terms in the above sum are equal, and we obtain

$$\mathbb{P}[\sigma^{(n),p} = \pi] = \frac{N_{\text{inc}}(\pi)}{(n-1)!} (1-p)^{\text{des}(\pi)} p^{n-1-\text{des}(\pi)}. \quad \square$$

4.3. Intensity measure. Recall that the intensity measure of μ_p^{rec} , denoted $I\mu_p^{\text{rec}}$, is a probability measure on $[0, 1]^2$ defined as follows: for all Borel subsets A of $[0, 1]^2$, we have $I\mu_p^{\text{rec}}(A) = \mathbb{E}[\mu_p^{\text{rec}}(A)]$. The goal of this section is to compute this intensity measure. We start with a lemma.

LEMMA 4.3. — *The intensity measure $I\mu_p^{\text{rec}}$ of the recursive separable permuton is the distribution of $(U, \phi(U))$, where U is a uniform random variable in $[0, 1]$, ϕ is the random function of Section 3.2, and U and ϕ are independent from each other.*

Proof. — For any function f , we denote f_2 the map $x \mapsto (x, f(x))$. Then, for any measurable set $A \subset [0, 1]^2$, we have

$$\mu_f(A) = \text{Leb}(f_2^{-1}(A)) = \int_0^1 \mathbf{1}[(x, f(x)) \in A] dx = \mathbb{E}[\mathbf{1}[(U, f(U)) \in A]],$$

where U is a uniform random variable in $[0, 1]$. Recalling that $\mu_p^{\text{rec}} = \mu_\phi$, where ϕ is the random function of Section 3.2, we have

$$I\mu_p^{\text{rec}}(A) = \mathbb{E}[\mu_\phi(A)] = \mathbb{E}[\mathbf{1}[(U, \phi(U)) \in A]],$$

where in the first expectation, only ϕ is random but, in the second one, both U and ϕ are random, independent from each other. This proves the lemma. \square

Remark 4.4. — A similar statement (proved in the same way) for the Brownian separable permuton is given in [20, Lemma 6.1], and serves as a basis of Maazoun's computation of the intensity measure of the Brownian separable permuton.

4.3.1. Distributional equations. To go further, we consider the following map Ψ_p from the set $\mathcal{M}_1([0, 1])$ of probability measures on $[0, 1]$ to itself. If ν is a probability measure on $[0, 1]$, we let X be a r.v. with distribution ν , and set

$$Y = V \cdot B + (1 - V) \cdot X,$$

where B is a Bernoulli variable with parameter p , V is a uniform random variable in $[0, 1]$, and the two are independent from each other and from X . We then define $\Psi_p(\nu)$ as the distribution of Y .

LEMMA 4.5. — *The equation $\nu = \Psi_p(\nu)$ has a unique solution in $\mathcal{M}_1([0, 1])$.*

This solution will be denoted ν_p from now on.

Proof. — We will prove that Ψ_p is a contracting map from $\mathcal{M}_1([0, 1])$ to itself, when we equip it with the first Wasserstein metric. Namely, we let ν_1 and ν_2 be two probability measures of $[0, 1]$ and we shall prove that

$$d_{W,1}(\Psi_p(\nu_1), \Psi_p(\nu_2)) \leq \frac{1}{2}d_{W,1}(\nu_1, \nu_2). \quad (4.5)$$

Fix $\varepsilon > 0$ and choose r.v. (X_1, X_2) on the same probability space such that

$$\mathbb{E}[|X_1 - X_2|] \leq d_{W,1}(\nu_1, \nu_2) + \varepsilon.$$

We then take a Bernoulli variable B with parameter p and a uniform random variable V in $[0, 1]$, independent from each other and from (X_1, X_2) . We set

$$Y_1 = V \cdot B + (1 - V) \cdot X_1, \quad Y_2 = V \cdot B + (1 - V) \cdot X_2.$$

Note that the variables V and B used to define Y_1 and Y_2 are the same. We have

$$\mathbb{E}[|Y_1 - Y_2|] \leq \mathbb{E}[|1 - V| \cdot |X_1 - X_2|] \leq \frac{1}{2}\mathbb{E}[|X_1 - X_2|] \leq \frac{1}{2}d_{W,1}(\nu_1, \nu_2) + \frac{1}{2}\varepsilon.$$

The random variables Y_1 and Y_2 are defined on the same probability space, and have distributions $\Psi_p(\nu_1)$ and $\Psi_p(\nu_2)$ respectively. By definition of the Wasserstein distance, we have

$$d_{W,1}(\Psi_p(\nu_1), \Psi_p(\nu_2)) \leq \mathbb{E}[|Y_1 - Y_2|] \leq \frac{1}{2}d_{W,1}(\nu_1, \nu_2) + \frac{1}{2}\varepsilon.$$

Since this holds for any $\varepsilon > 0$, we have proved Eq. (4.5). We conclude with Banach fixed point theorem that Ψ_p has exactly one fixed point, concluding the proof of the lemma. \square

PROPOSITION 4.6. — *The intensity measure $I\mu_p^{\text{rec}}$ of the recursive separable permuton is the distribution of*

$$(U, UX_p + (1 - U)X_{1-p}),$$

where U , X_p and X_{1-p} are independent r.v. with distribution $\text{Leb}([0, 1])$, ν_p and ν_{1-p} distribution respectively.

Proof. — From Lemma 4.3, $I\mu_p^{\text{rec}}$ is the distribution of $(U, \phi(U))$, where U is a uniform random variable in $[0, 1]$ and ϕ the random mapping constructed in Section 3.2. We write $\phi(U) = Y_1 + Y_2$, with

$$\begin{cases} Y_1 = \text{Leb}(\{y : (y < U) \wedge (y \prec U)\}); \\ Y_2 = \text{Leb}(\{y : (y > U) \wedge (y \prec U)\}); \end{cases}$$

In each of these equations, the first comparison is for the natural order on $[0, 1]$, while the second is for the random order \prec .

We shall describe the distribution of Y_2 , conditionally on U . Let V_1 be the first U_i larger than U , “first” meaning here the one with smallest index. We then denote V_2 to be the first U_i between U and V_1 and define $V_3, V_4 \dots$ similarly. Clearly, setting for convenience $V_0 = 1$, we have

$$(V_{j+1} - U)_{j \geq 0} \stackrel{d}{=} (T_{j+1} (V_j - U))_{j \geq 0},$$

where (T_1, T_2, \dots) is a sequence of i.i.d. uniform random variable in $[0, 1]$, independent from U . This implies

$$(V_j)_{j \geq 0} \stackrel{d}{=} (U + (1 - U)T_1 \cdots T_j)_{j \geq 0}.$$

Each V_j inherits a sign, i.e. if $V_j = U_i$, we set $\Sigma_j = S_i$. By construction, if $\Sigma_j = \ominus$, every y in the interval $[V_j, V_{j-1}]$ satisfies $y < U$. On the other hand, $\Sigma_j = \oplus$, every y in the interval $[V_j, V_{j-1}]$ satisfies $y > U$. Hence

$$Y_2 = \sum_{j \geq 1: \Sigma_j = \ominus} (V_{j-1} - V_j) = (1 - U) \sum_{j \geq 1: \Sigma_j = \ominus} T_1 \cdots T_{j-1} (1 - T_j).$$

Setting $\tilde{Y}_2 = Y_2 / (1 - U)$ and interpreting \oplus as 1 and \ominus as 0, we have

$$\tilde{Y}_2 = (1 - \Sigma_1)(1 - T_1) + T_1 \left(\sum_{j \geq 2: \Sigma_j = \ominus} T_2 \cdots (1 - T_j) \right).$$

Note that \tilde{Y}_2 is independent from U . Moreover, the variable $1 - \Sigma_j$ is a Bernoulli r.v. of parameter $1 - p$, while T_1 is uniform in $[0, 1]$. Finally, the sum in parentheses has the same distribution as \tilde{Y}_1 , and is independent from Σ_1 and T_1 . This shows that the distribution of \tilde{Y}_2 is a fixed point of Ψ_{1-p} . From Lemma 4.5, \tilde{Y}_2 has distribution ν_{1-p} .

With similar arguments, one can show that $Y_1 = U\tilde{Y}_1$, where \tilde{Y}_1 has distribution ν_p and is independent from U (and from \tilde{Y}_2). This ends the proof of the proposition. \square

4.3.2. Explicit formulas for densities.

LEMMA 4.7. — *The unique solution ν_p of the equation $\nu_p = \Psi(\nu_p)$ is the beta distribution of parameters $(p, 1 - p)$. Explicitly it is given by*

$$\nu_p(dx) = \frac{1}{\Gamma(p)\Gamma(1-p)} x^{p-1}(1-x)^{-p} dx,$$

where Γ is the usual gamma function.

Proof. — Let X be a random variable with distribution $\text{Beta}(p, 1 - p)$, and set $Y = V \cdot B + (1 - V)X$, where B and V are as above. We want to show that $Y \stackrel{d}{=} X$, which would imply the distribution of X is a fixed point of Ψ as wanted.

Let f be a continuous function on $[0, 1]$. Setting $Z = \Gamma(p)\Gamma(1-p)$, we have

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{p}{Z} \int_{[0,1]^2} f(v + (1-v)x) x^{p-1}(1-x)^{-p} dv dx \\ &\quad + \frac{1-p}{Z} \int_{[0,1]^2} f((1-v)x) x^{p-1}(1-x)^{-p} dv dx \\ &= \frac{p}{Z} \int_0^1 f(u) \left(\int_0^u x^{p-1}(1-x)^{-p-1} dx \right) du \\ &\quad + \frac{1-p}{Z} \int_0^1 f(u) \left(\int_u^1 x^{p-2}(1-x)^{-p} dx \right) du. \end{aligned} \tag{4.6}$$

(In the first integral, we have performed the change of variables $u = v + (1 - v)x$, yielding $dv = \frac{dx}{1-x}$; in the second, we set $u = (1 - v)x$, yielding $dv = \frac{du}{x}$.)

We claim that there exists a constant A in \mathbb{R} , such that, for every u in $(0, 1)$,

$$p \int_0^u x^{p-1}(1-x)^{-p-1} dx + (1-p) \int_u^1 x^{p-2}(1-x)^{-p} dx = u^{p-1}(1-u)^{-p} + A.$$

Indeed, one checks easily that both sides have the same derivative. With this equality in hand, (4.6) rewrites as

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{1}{Z} \int_0^1 f(u) u^{p-1} (1-u)^{p-1} du + \frac{A}{Z} \int_0^1 f(u) du \\ &= \mathbb{E}[f(X)] + \frac{A}{Z} \int_0^1 f(u) du. \end{aligned}$$

Choosing for f the function constant equal to 1 shows that necessarily $A = 0$. Thus we have that $\mathbb{E}[f(Y)] = \mathbb{E}[f(X)]$ for any continuous function f of $[0, 1]$, implying that X and Y have the same distribution. This ends the proof of the Lemma 4.7. \square

Proposition 4.6 and Lemma 4.7 imply Proposition 1.9. It remains to prove Corollary 1.10.

Proof of Corollary 1.10. Let f be a continuous function from $[0, 1]^2$ to \mathbb{R} . From Proposition 1.9, we have

$$\begin{aligned} \Gamma(p)^2 \Gamma(1-p)^2 \int f(x, y) I\mu_p^{\text{rec}}(dx, dy) \\ = \int_{[0,1]^3} f(u, ua + (1-u)b) du a^{p-1}(1-a)^{-p} da b^{-p}(1-b)^{p-1} db \end{aligned}$$

We perform the change of variable

$$x = u; \quad y = ua + (1-u)b; \quad z = ua$$

This maps bijectively the (u, a, b) -domain $[0, 1]^3$ to the set

$$\{(x, y, z) : \max(x + y - 1, 0) \leq z \leq \min(x, y)\}.$$

The Jacobian matrix of the transformation is

$$J := \frac{\partial(x, y, z)}{\partial(u, a, b)} = \begin{pmatrix} 1 & 0 & 0 \\ a-b & u & 1-u \\ a & u & 0 \end{pmatrix},$$

whose determinant satisfies $|\det(J)| = u(1-u) = x(1-x)$. Therefore we have

$$\begin{aligned} \Gamma(p)^2 \Gamma(1-p)^2 \int f(x, y) I\mu_p^{\text{rec}}(dx, dy) &= \int_{[0,1]^2} \left[f(x, y) \cdot \right. \\ &\quad \left. \cdot \left(\int_{\max(x+y-1,0)}^{\min(x,y)} \left(\frac{z}{x}\right)^{p-1} (1-\frac{z}{x})^{-p} \left(\frac{y-z}{1-x}\right)^{-p} \left(1-\frac{y-z}{1-x}\right)^{p-1} dz \right) \frac{dx dy}{x(1-x)} \right]. \end{aligned}$$

After elementary simplifications, we get

$$\begin{aligned} \Gamma(p)^2 \Gamma(1-p)^2 \int f(x, y) I\mu_p^{\text{rec}}(dx, dy) &= \int_{[0,1]^2} \left[f(x, y) \right. \\ &\quad \left. \cdot \left(\int_{\max(x+y-1,0)}^{\min(x,y)} \frac{dz}{z^{1-p}(x-z)^p(y-z)^p(1-x-y+z)^{1-p}} \right) dx dy \right]. \end{aligned}$$

This proves Corollary 1.10. \square

4.4. Mutual singularity of the limiting permutons. In this section, we prove Proposition 1.3 in two independent steps. First, we prove the singularity of separable Brownian or recursive separable permutons associated with different values p and q of the parameter. Then we compare specifically Brownian and recursive separable permutons μ_p^{rec} and μ_p^{Br} , associated with the same value p of the parameter.

Before starting the proof, let us recall, more formally than in Remark 3.1, the construction of μ_p^{Br} , as given in [20]. We start with a Brownian excursion ϵ on $[0, 1]$ and a sequence $(S_m)_{m \in \text{Min}(\epsilon)}$ of signs indexed by local minima of ϵ . Conditionally on ϵ , the variables $(S_m)_{m \in \text{Min}(\epsilon)}$ are i.i.d. with distribution

$$\mathbb{P}(S_m = \oplus) = p = 1 - \mathbb{P}(S_m = \ominus).$$

Given such a sequence we define a partial order \prec_{Br} as follows: for $x < y$ in $[0, 1]$, we let m be the position of the minimum of ϵ on the interval $[x, y]$ and set

$$\begin{cases} x \prec_{\text{Br}} y & \text{if } S_m = \oplus; \\ y \prec_{\text{Br}} x & \text{if } S_m = \ominus. \end{cases}$$

Note that, if $m \in \{x, y\}$, then m might not be a local minimum, in which case S_m is ill-defined and x and y are incomparable by convention. This happens only for a measure 0 subset of pairs (x, y) (w.r.t. Lebesgue measure).

The rest of the construction is then similar to that of the recursive separable permuton: we define

$$\phi_{\text{Br}}(x) = \text{Leb}(\{y \in [0, 1] : y \prec_{\text{Br}} x\}),$$

and let μ_p^{Br} be the push-forward of the Lebesgue measure on $[0, 1]$ by the map $x \mapsto (x, \phi_{\text{Br}}(x))$.

4.4.1. Comparing permutons with different values of p . We recall from Section 1.3.2 that given a permuton μ and an integer $n \geq 1$, we can define a random permutation $\text{Sample}(\mu, n)$ by sampling independent points according to μ . Also, for a permutation π , we let $\text{des}(\pi)$ be its number of descents. We start with a lemma.

LEMMA 4.8. — *Let p be in $(0, 1)$. Then the random variable*

$$D_n = D_n(\mu_p^{\text{rec}}) := \frac{1}{n-1} \mathbb{E} [\text{des}(\text{Sample}(\mu_p^{\text{rec}}, n)) \mid \mu_p^{\text{rec}}].$$

converges to $1 - p$ a.s. Moreover, the same holds replacing μ_p^{rec} by μ_p^{Br} .

Proof. — We have

$$\mathbb{E}(D_n) = \frac{1}{n-1} \mathbb{E} [\text{des}(\text{Sample}(\mu_p^{\text{rec}}, n))].$$

From the proof of Proposition 1.7, the permutation $\text{Sample}(\mu_p^{\text{rec}}, n)$ has the same distribution as the permutation represented by a uniform random increasing binary tree T_n , where each internal node is decorated with \oplus independently with probability p . As already observed, if π is encoded by a binary decorated tree T , then $\text{des}(\pi)$ is the number of \ominus signs in T . Hence $\text{des}(\text{Sample}(\mu_p^{\text{rec}}, n))$ is the number of minus signs in T_n , and its law is that of a binomial random variable $\text{Bin}(n-1, 1-p)$. We deduce that $\mathbb{E}(D_n) = 1 - p$.

To get a.s. convergence, we consider the fourth centered moment of D_n . We have, using Jensen's inequality for conditional expectation,

$$\begin{aligned} \mathbb{E}[(D_n - (1-p))^4] &= \mathbb{E}\left[\mathbb{E}\left(\frac{1}{n-1} \text{des}(\text{Sample}(\boldsymbol{\mu}_p^{\text{rec}}, n)) - (1-p) \mid \boldsymbol{\mu}_p^{\text{rec}}\right)^4\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\left(\frac{1}{n-1} \text{des}(\text{Sample}(\boldsymbol{\mu}_p^{\text{rec}}, n)) - (1-p)\right)^4 \mid \boldsymbol{\mu}_p^{\text{rec}}\right]\right] \\ &= \mathbb{E}\left[\left(\frac{1}{n-1} \text{des}(\text{Sample}(\boldsymbol{\mu}_p^{\text{rec}}, n)) - (1-p)\right)^4\right]. \end{aligned}$$

The expectation in the right-hand side is the centered fourth moment of the average of $n-1$ independent Bernoulli random variables of parameter $1-p$, which is known to behave as $O(n^{-2})$. We get that

$$\mathbb{E}[(D_n - (1-p))^4] = O(n^{-2}),$$

and hence it is a summable quantity. By a classical application of Borel–Cantelli lemma, this implies that D_n converges a.s. to $1-p$.

The proof for the Brownian separable permuton is similar. Indeed, from [20, Definition 2], we know that the quantity $\text{des}(\text{Sample}(\boldsymbol{\mu}_p^{\text{Br}}, n))$ is also distributed as a binomial random variable $\text{Bin}(n-1, 1-p)$. \square

COROLLARY 4.9. — *If $p \neq q$ are in $(0, 1)$, then $\boldsymbol{\mu}_p^{\text{rec}}$ and $\boldsymbol{\mu}_q^{\text{rec}}$ are mutually singular. The same holds, replacing either $\boldsymbol{\mu}_p^{\text{rec}}$ or $\boldsymbol{\mu}_q^{\text{rec}}$, or both, by $\boldsymbol{\mu}_p^{\text{Br}}$ or $\boldsymbol{\mu}_q^{\text{Br}}$.*

Proof. — Let E_p be the set of permutons μ such that

$$D_n(\mu) = \frac{1}{n-1} \mathbb{E}[\text{des}(\text{Sample}(\mu, n))]$$

converges to $1-p$ (we do not take a conditional expectation here, since μ is deterministic). By Lemma 4.8, $\boldsymbol{\mu}_p^{\text{rec}}$ and $\boldsymbol{\mu}_p^{\text{Br}}$ belongs to E_p a.s. Since E_p and E_q are disjoint for $p \neq q$, the corollary follows. \square

4.4.2. Comparing the Brownian and recursive separable permutons with the same parameter. We now want to prove that $\boldsymbol{\mu}_p^{\text{rec}}$ and $\boldsymbol{\mu}_p^{\text{Br}}$ are singular. Since both $D_n(\boldsymbol{\mu}_p^{\text{rec}})$ and $D_n(\boldsymbol{\mu}_p^{\text{Br}})$ converge to the same value $1-p$ a.s., we need to find another distinguishing feature. We will prove that in the Brownian separable permuton, all corners of the square a.s. carry some mass, which is not the case for the recursive separable permuton.

To this end, for $\varepsilon > 0$, we introduce the following events (we recall that \mathcal{P} is the set of permutons):

$$\begin{aligned} \text{BL}_\varepsilon &= \{\mu \in \mathcal{P} : \mu([0, \varepsilon]^2) > 0\} \\ \text{TL}_\varepsilon &= \{\mu \in \mathcal{P} : \mu([0, \varepsilon] \times [1-\varepsilon, 1]) > 0\}. \end{aligned}$$

In words, BL_ε is the set of permutons that have some mass in an ε -neighbourhood of the bottom-left corner; TL_ε is the same using the top-left corner.

We now prove two lemmas, showing that the Brownian and recursive separable permutons behave differently with respect to these events. We start with the recursive separable permuton.

LEMMA 4.10. — *Fix $p \in (0, 1)$ and $\varepsilon > 0$. We have*

$$\mathbb{P}(\boldsymbol{\mu}_p^{\text{rec}} \in \text{BL}_\varepsilon \cap \text{TL}_\varepsilon) \leq 2\varepsilon.$$

Proof. — We recall the construction of μ_p^{rec} from Section 3.2, and use the notation introduced there.

We temporarily assume that U_1 is in $(\varepsilon, 1 - \varepsilon)$ and that $S_1 = \oplus$. We consider some $x < \varepsilon$. For any $y \geq U_1$, we have $U_1 \in (x, y]$, and therefore $i_{x,y} = 1$ with the notation of Section 3.2. Since $x < y$ and $S_1 = \oplus$, this implies $x \prec y$. Taking the contraposition, $y \prec x$ implies $y < U_1$. Therefore,

$$\phi(x) = \text{Leb}(\{y \in [0, 1] : y \prec x\}) \leq \text{Leb}(\{y \in [0, 1] : y < U_1\}) \leq U_1 < 1 - \varepsilon.$$

Thus there does not exist x such that $(x, \phi(x)) \in [0, \varepsilon] \times [1 - \varepsilon, 1]$. Consequently, under the assumptions $U_1 \in (\varepsilon, 1 - \varepsilon)$ and $S_1 = \oplus$, we have

$$\mu_p^{\text{rec}}([0, \varepsilon] \times [1 - \varepsilon, 1]) = 0, \text{ or equivalently, } \mu_p^{\text{rec}} \notin \text{TL}_\varepsilon.$$

Similarly, one can prove that, if U_1 is in $(\varepsilon, 1 - \varepsilon)$ and $S_1 = \ominus$, then $\mu_p^{\text{rec}} \notin \text{BL}_\varepsilon$. Therefore we have

$$\mathbb{P}(\mu_p^{\text{rec}} \in \text{BL}_\varepsilon \cap \text{TL}_\varepsilon) \leq \mathbb{P}(U_1 \notin (\varepsilon, 1 - \varepsilon)) \leq 2\varepsilon. \quad \square$$

Considering the Brownian separable permutation instead, we have the following.

LEMMA 4.11. — *Fix $p \in (0, 1)$ and $\varepsilon > 0$. We have*

$$\mathbb{P}(\mu_p^{\text{Br}} \in \text{BL}_\varepsilon) = 1.$$

Proof. — As recalled above, μ_p^{Br} can be constructed starting from a Brownian excursion \mathbf{e} and a sequence of signs $(S_m)_{m \in \text{Min}(\mathbf{e})}$ indexed by the (positions of) the local minima of \mathbf{e} . We fix a realization of \mathbf{e} and $\mathbf{S} = (S_m)_{m \in \text{Min}(\mathbf{e})}$ (and hence of μ_p^{Br}); most quantities below, including the random order \prec_{Br} and the function ϕ_{Br} , depend implicitly on \mathbf{e} and \mathbf{S} . Our first goal is to find a local minimum m_0 of \mathbf{e} , such that $S_{m_0} = \oplus$, $m_0 < \varepsilon/2$ and $\mathbf{e}(m_0) < \min_{t \in [m_0, 1 - \varepsilon/2]} \mathbf{e}(t)$. Since the proof involves quite a bit of notation, we illustrate it on Figure 4.2.

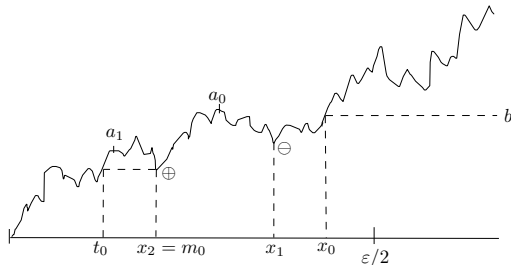


FIGURE 4.2. Illustration of the notation involved in the proof of Lemma 4.11. For readability, we have only represented an initial segment of the Brownian excursion \mathbf{e} . In the picture, we have $S_{x_1} = \ominus$ so that we have to find another candidate x_2 as explained in the proof. This time we have $S_{x_2} = \oplus$ and we set $m_0 = x_2$.

We first let $b = \min_{t \in [\varepsilon/2, 1 - \varepsilon/2]} \mathbf{e}(t)$ and $x_0 = \sup\{t < \varepsilon/2 : \mathbf{e}(t) = m\}$. A.s., we can then find $a_0 < x_0$ such that \mathbf{e} reaches its minimum on $[a_0, x_0]$ somewhere in the interior of this interval (if it was not the case, \mathbf{e} would be increasing on an initial segment $[0, \delta]$ for some δ , which is known to happen with probability 0). Let x_1 be the point where \mathbf{e} is minimal on $[a_0, x_0]$. This is a local minimum, and therefore

carries a sign S_{x_1} . If $S_{x_1} = \oplus$, we define $m_0 = x_1$, and we verify easily that it satisfies the desired properties.

Otherwise, we iterate the process: a.s., we can then find $a_1 < x_1$ such that \mathbf{e} reaches its minimum on $[a_1; x_1]$ somewhere in the interior of this interval. We call x_2 the point where this minimum is reached. If $S_{x_2} = \oplus$, we define $m_0 = x_2$. If not, we iterate another time. Doing so, we will construct a sequence of local minima x_1, x_2, \dots , and since the associated signs are i.i.d. and equal to \oplus with positive probability, we will eventually find x_i with $S_{x_i} = \oplus$. Then we set $m_0 = x_i$, and verify easily that m_0 satisfies the desired properties.

Having found m_0 , we look for the last time t_0 before m_0 with $\mathbf{e}(t_0) = \mathbf{e}(m_0)$. By construction, if we take x in (t_0, m_0) and y in $(m_0, 1 - \varepsilon/2)$ the minimum of \mathbf{e} in the interval $[x, y]$ is reached in m_0 . Since $S_{m_0} = \oplus$, we have $x \prec_{\text{Br}} y$, where \prec_{Br} is the order appearing in the construction of the Brownian separable permuton. Therefore, for x in (t_0, m_0) , we have

$$\phi_{\text{Br}}(x) = \text{Leb}(\{y \in [0, 1] : y \prec_{\text{Br}} x\}) \leq 1 - \text{Leb}((m_0, 1 - \varepsilon/2)) = m_0 + \varepsilon/2 < \varepsilon.$$

Letting U be a uniform random variable in $[0, 1]$, we have, a.s.,

$$\begin{aligned} \mu_p^{\text{Br}}([0, \varepsilon]^2) &= \mathbb{P}((U, \phi_{\text{Br}}(U)) \in [0, \varepsilon]^2 | (\mathbf{e}, \mathbf{S})) \geq \mathbb{P}(U \in (t_0, m_0) | (\mathbf{e}, \mathbf{S})) \\ &= m_0 - t_0 > 0; \end{aligned}$$

(recall that m_0 and t_0 depends on (\mathbf{e}, \mathbf{S}) and thus are random variables themselves). This proves the lemma. \square

COROLLARY 4.12. — *Fix p in $(0, 1)$. Then the distributions of μ_p^{rec} and μ_p^{Br} are mutually singular.*

Proof. — We consider the decreasing intersection

$$E := \bigcap_{\varepsilon > 0} (\text{BL}_\varepsilon \cap \text{TL}_\varepsilon).$$

From Lemma 4.10, we know that $\mathbb{P}(\mu_p^{\text{rec}} \in E) = 0$. On the other hand, Lemma 4.11 tells us that, for any $\varepsilon > 0$, we have

$$\mathbb{P}(\mu_p^{\text{Br}} \in \text{BL}_\varepsilon) = 1.$$

By symmetry, the same holds replacing BL_ε by TL_ε , and thus, for any $\varepsilon > 0$,

$$\mathbb{P}(\mu_p^{\text{Br}} \in \text{BL}_\varepsilon \cap \text{TL}_\varepsilon) = 1.$$

Consequently, $\mathbb{P}(\mu_p^{\text{rec}} \in E) = 1$, proving the corollary. \square

Remark 4.13. — The difference between recursive and Brownian separable permutons which we exhibited in the proof can be observed on simulations. Figure 4.3 shows simulations of a random recursive separable permutation with parameter $p = 1/2$ (on the left) and of a uniform random separable permutation (on the right), both of size 1000. From the result of this paper and of [5], the corresponding limiting permutons are the recursive and Brownian separable permutons, respectively, each time of parameter $p = 1/2$. We see that in the Brownian case (picture on the right), there are some points relatively close to each one of the four corners of the square, which is not the case in the recursive case (picture on the left).

An informal explanation of that can also be given at the discrete level. As we have seen, recursive separable permutations of size n are associated with uniform

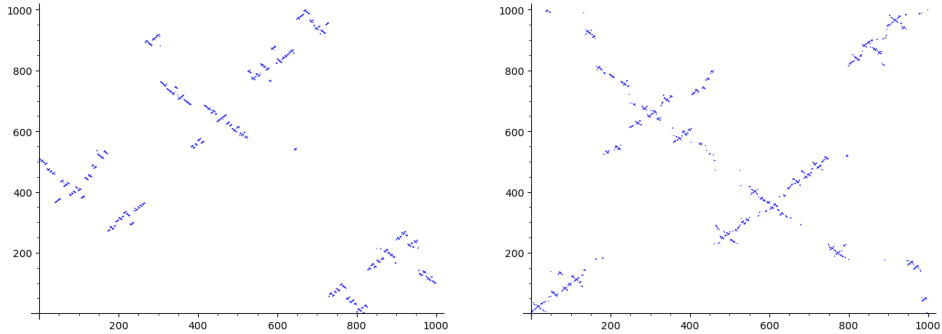


FIGURE 4.3. Diagram of random separable permutations of 1000: on the left, the permutation has been generated via the recursive algorithm given in Section 1.1; on the right it is a uniform random separable permutation of size 1000 (sampled via the so-called recursive method).

random *labelled increasing* binary trees (Section 4.2). It is known that in such random trees, the two subtrees attached to the root splits both have macroscopic size (this follows, for example, from the correspondence between increasing trees and permutations; see, e.g., [15, Example II.7]). Hence $\sigma^{(n),p} = \tau \oplus \rho$ or $\sigma^{(n),p} = \tau \ominus \rho$, for some permutations τ and ρ of macroscopic sizes (this can also directly be seen on Proposition 4.2). This explains why some corners of the permutation diagram of $\sigma^{(n),p}$ are empty.

On the other hand, uniform random separable permutations of size n are associated with a uniform random Schrodler tree (see [5]). In such a random tree, among the subtree attached to the root, a single one of them contains most of the vertices of the trees, the other ones having typically size $O(1)$; see, e.g., the local limit results given in [19]. Such unbalanced splits are repeated many times, with signs either \oplus and \ominus , before a macroscopic split arises, and this explains why we find points in the permutation diagram near all corners of the square.

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Finally, simulations of Figures 1.2, 1.8, 3.2, 3.3 and 4.3, as well as the generation of data for Figure 1.7, were done with the computer algebra software SageMath [25]. The authors are grateful to its developers for their wonderful job.

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